Defn: Let $f : \mathbf{R}^n \to \mathbf{R}^m$.

$$\frac{\partial f_i}{\partial x_j}(\mathbf{a}) = lim_{h \to 0} \frac{f_i(a_1, \ldots, a_{j-1}, a_j + h, a_{j+1}, \ldots, a_n) - f_i(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n)}{h}$$

$$= lim_{h \to 0} \frac{f_i(\mathbf{a} + h\mathbf{e_j}) - f_i(\mathbf{a})}{h}$$

Ex: $f : \mathbf{R}^2 \to \mathbf{R}$, $f(x, y) = \begin{cases} 0 & (x, y) = (0, 0) \\ \frac{xy}{x^2 + y^2} & \text{otherwise} \end{cases}$

$$\frac{\partial f}{\partial x}(0,0) = lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = lim_{h \to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h} = lim_{h \to 0} \frac{0-0}{h} = 0$$

f not differentiable

BUT f is not continuous at (0, 0)!!!!!!!!!

Defn: Suppose $f : \mathbf{R}^1 \to \mathbf{R}^1$ is differentiable at a. Then

$$lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

$$lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

$$lim_{x \to a} \frac{f(x) - f(a)}{x - a} - f'(a) = 0$$

$$lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x-a)}{(x-a)} = 0$$

$$lim_{x \to a} \frac{f(x) - [f(a) + f'(a)(x-a)]}{(x-a)} = 0$$

$y = f'(a)[x - a] + f(a)$ is the linear approximation of f near a.

$\boxed{y = f(a) + f'(a)(x-a)}$ $\Leftarrow$ tangent line

$\approx f(x)$ near a

Defn: The *gradient of* f is denoted by

$$\nabla f(\mathbf{a}) = \left(\frac{\partial f_1}{\partial x_1}(\mathbf{a}), ..., \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \right)$$

Defn: The *Jacobian matrix of* f *at* a is

$$Df(\mathbf{a}) = \left(\tfrac{\partial f_i}{\partial x_j}(\mathbf{a}) \right)_{m \times n} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{pmatrix}$$

Defn: Suppose $f : \mathbf{R}^1 \to \mathbf{R}^1$ is differentiable at a. Then

$$lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0$$

$$lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x-a)}{(x-a)} = 0$$

Defn: Suppose $A \subset \mathbf{R}^n$, $f : A \to \mathbf{R}^m$.

f is said to be **differentiable at a point a** if there exists an open ball V such that $a \in V \subset A$ and a linear function T such that that

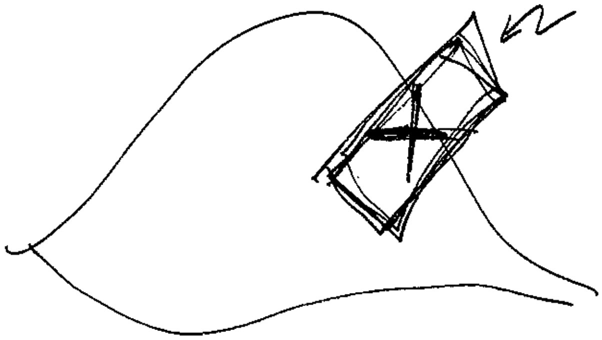
$$lim_{\mathbf{h} \to \mathbf{0}} \frac{||f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - T(\mathbf{h})||}{||\mathbf{h}||} = 0 \qquad \boxed{T = Df(\vec{a})}$$

$$lim_{\mathbf{x} \to \mathbf{a}} \frac{||f(\mathbf{x}) - f(\mathbf{a}) - T(\mathbf{x}-\mathbf{a})||}{||\mathbf{x}-\mathbf{a}||} = 0$$

near $\vec{a}$

$f(\vec{x}) \approx f(\vec{a}) + T^3(\vec{x} - \vec{a})$ ← tangent hyperplane ← approximate f near $\vec{a}$

tangent plane

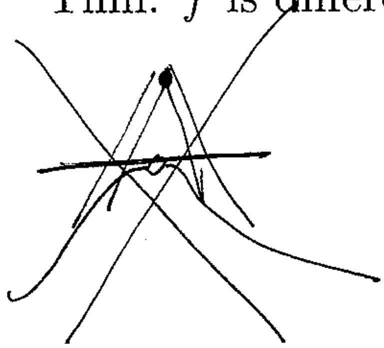


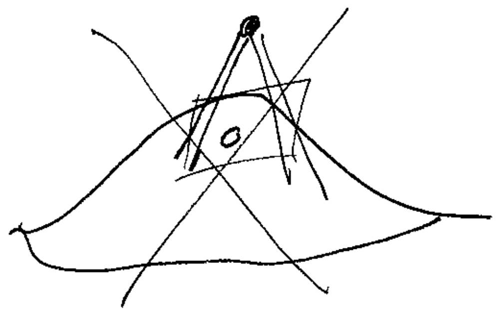
$f : \mathbb{R}^2 \to \mathbb{R}$

$$\boxed{f(\vec{x}) \approx f(\vec{a}) + Df(\vec{a})\,(\vec{x}-\vec{a})}$$

tangent (hyper) plane

Thm: f is differentiable at $\mathbf{a}$ implies f is continuous at $\mathbf{a}$.





Thm: Let $f : \mathbf{R}^n \to \mathbf{R}^m$, $f = (f_1, ..., f_m)$. f is differentiable at $\mathbf{a}$ iff $f_i : \mathbf{R}^n \to \mathbf{R}$ is differentiable at $\mathbf{a}$ for all $i = 1, ..., m$

Thm: If f is differentiable at $\mathbf{a}$ then $\frac{\partial f_i}{\partial x_j}$ exists for all i, j and $Df(\mathbf{a})$ = the Jacobian evaluated at $\mathbf{a}$.

If Jacobian exists $\not\Rightarrow$ f differentiable

Thm: Let $f : \mathbf{R}^n \to \mathbf{R}^m$, $f = (f_1, ..., f_m)$. If $\frac{\partial f_i}{\partial x_j}$ exists and are continuous in a neighborhood of $\mathbf{a}$ for all i, j, then f is differentiable at $\mathbf{a}$

Ex: Is $f(x, y) = x^2 y$ differentiable at (3, 1)? Yes

$Df(x,y) = (2xy,\ x^2)$

Let $g(x,y) = 2xy$

$\frac{\partial f}{\partial x}(x,y) = 2xy$ is cont

$\frac{\partial f}{\partial y}(x,y) = x^2$ is cont

$\Rightarrow f$ diff

Find the equation of the tangent plane to $f(x, y) = x^2 y$ at (3, 1).

Estimate f(3.1, .9)

tangent plane $f(x,y)=x^2y$
at $(3,1)$

$$z = f(\vec{a}) + Df(\vec{a})(\vec{x}-\vec{a})$$

$$Df(x,y) = (2xy,\ x^2)$$

$$Df(3,1) = (6, 9)$$

$$z = f(3,1) + (6,9)\left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right]$$

$$= f(3,1) + (6,9)\begin{pmatrix} x-3 \\ y-1 \end{pmatrix}$$

$$= 9 + 6(x-3) + 9(y-1)$$

$$= 9 + 6x - 18 + 9y - 9$$

$$\boxed{z = 6x + 9y - 18}$$ eqn of tangent plane

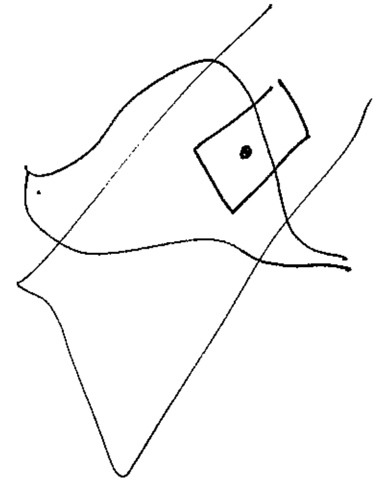
approx mate f near a

$f(3.1, .9) \approx$

$6(3.1) + 9(.9) - 18$

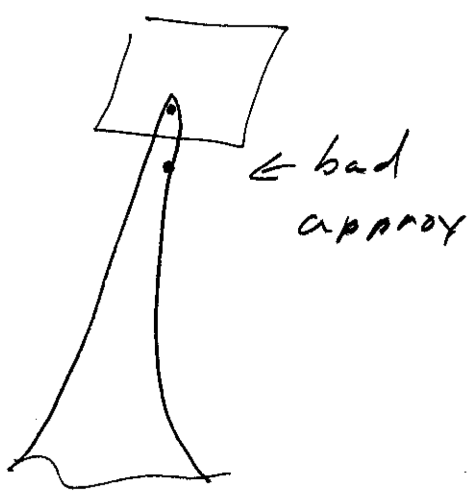
$= 18.6 + 8.1 - 18$

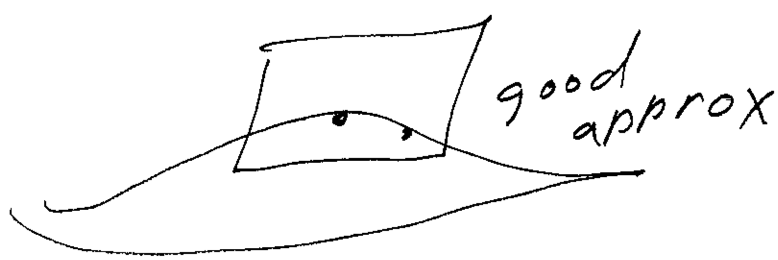
$= 8.7$



$f(x,y) = x^2 y$

$f(3.1, .9) = (3.1)^2(.9) = 8.649$





2.4

Thm: If $f, g : \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at $\mathbf{a}$, then $f + g$ is differentiable at $\mathbf{a}$ and $D(f+g) = Df + Dg$.

Thm: Let $c \in \mathbf{R}$. If $f : \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at $\mathbf{a}$, then cf is differentiable at $\mathbf{a}$ and $D(cf) = cDf$.

$f(g(x))$

2.5 Thm: If $g : \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at $\mathbf{a}$ and if $f : \mathbf{R}^m \to \mathbf{R}^k$ is differentiable at g($\mathbf{a}$), then $f \circ g$ is differentiable at $\mathbf{a}$ and $D(f \circ g)(\mathbf{a}) = Df(g(\mathbf{a}))Dg(\mathbf{a})$.

matrix multiplication

Note for the product and quotient rule, f, g are real-valued functions, NOT vector valued.

$f : \mathbb{R}^n \to \mathbb{R}^m$ $\quad f_i : \mathbb{R}^n \to \mathbb{R}^1$

Thm: If $f, g : \mathbf{R}^n \to \mathbf{R}$ is differentiable at $\mathbf{a}$, then fg is differentiable at $\mathbf{a}$ and $D(fg) = g(\mathbf{a})Df(\mathbf{a}) + f(\mathbf{a})Dg(\mathbf{a})$.

vector $\quad$ vector

Thm: If $g(\mathbf{a}) \neq 0$ and $f, g : \mathbf{R}^n \to \mathbf{R}$ is differentiable at $\mathbf{a}$, then f/g is differentiable at $\mathbf{a}$ and $D(f/g) = \frac{g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})}{g(\mathbf{a})^2}$.

$f(x,y) = (x^2, y)$

$g(x,y) = (\boxed{y \ln x}, 5)$

$f(x,y) + g(x,y) = (x^2 + y \ln x, y + 5)$

$$Df(x,y) = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix}$$

$$Dg(x,y) = \begin{pmatrix} y/x & \ln x \\ 0 & 0 \end{pmatrix}$$

$$D(f+g) = \begin{pmatrix} 2x + y/x & 0 + \ln x \\ 0 + 0 & 1 + 0 \end{pmatrix}$$

$$D(5f) = 5\,Df = 5\begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 10x & 0 \\ 0 & 5 \end{pmatrix}$$

$5f(x,y) = (5x^2, 5y)$

$$f(x,y) = \left(\sin(x+y),\ x^2y^3\right)$$

$$g(x,y) = \left(e^{xy},\ \ln(x^2+y)\right)$$

$$fg(x,y) = \left(\sin(x+y)\,e^{xy},\ x^2y^3\ln(x^2+y)\right)$$

$$Dfg(x,y) = \begin{pmatrix} \cos(x+y)e^{xy} + ye^{xy}\sin(x+y) \\ \\ \\ \\ \end{pmatrix}$$

$$Df_g(x,y) = \begin{pmatrix} \cos(x+y)\,e^{xy} + y e^{xy} \sin(x+y) & \cos(x+y)\,e^{xy} + \sin(x+y)\,e^{xy}(x \\ 2xy^3 \ln(x^2+y) + x^2y^3 \frac{2x}{x^2+y} & 3x^2y^2 \ln(x^2+y) + x^2y^3 \frac{1}{x^2+y}, \end{pmatrix}$$

$$(fg)_1 = \sin(x+y)\, e^{xy}$$

$$f_1(x,y) = \sin(x+y)$$
$$g_1(x,y) = e^{xy}$$

$$D\big((fg)_1\big) = g(x,y)\, \underline{Df(x,y)} + f(x,y)\, Dg(x,y)$$
$$= e^{xy}\big(\cos(x+y), \cos(x+y)\big)$$
$$+ \sin(x+y)\,\big(ye^{xy}, xe^{xy}\big)$$

$$=$$