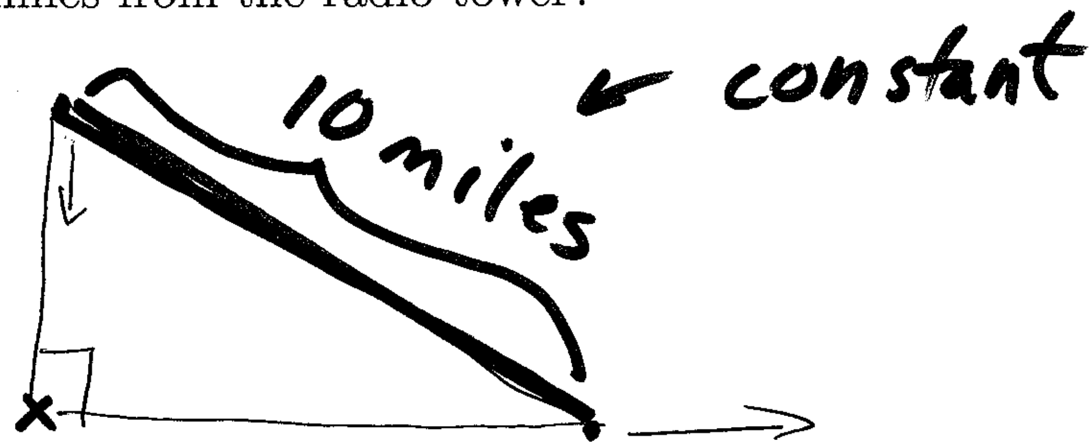
2.) Suppose the distance between two planes must be maintained at 10 miles. Suppose plane W is north of a radio tower and moving south while plane G is east of the same radio tower. If plane G is moving east at 1 mile/second, how fast should plane W be moving when plane G is 6 miles from the radio tower?

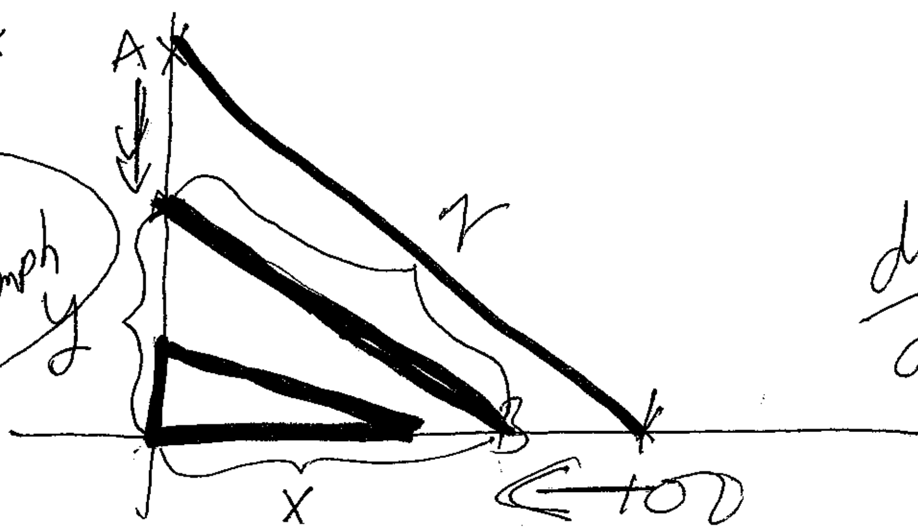


3.7) Related Rates

2.) Suppose car A is 110 miles north of an intersection and traveling south at 50 mph. Suppose car B is 100 miles east of the same intersection and traveling west at 20 mph. 1a.) At what rate are the cars approaching each other after 1 hour? 1b.) After 3 hours?

0) Draw picture

$\frac{dy}{dt} = -50\text{mph}$



$\frac{dx}{dt} = -20\text{mph}$

1) What are we looking for

$$\frac{dr}{dt}$$

2) Equation involving r

$$\boxed{x^2 + y^2 = r^2}$$

3,4) Simplify, take derivative
simplify, solve

$$\frac{d}{dt}\left[x^2+y^2\right] = \frac{d}{dt}\left[r^2\right]$$

$$\cancel{2}x\cdot\frac{dx}{dt} + \cancel{2}y\cdot\frac{dy}{dt} = \cancel{2}r\cdot\frac{dr}{dt}$$

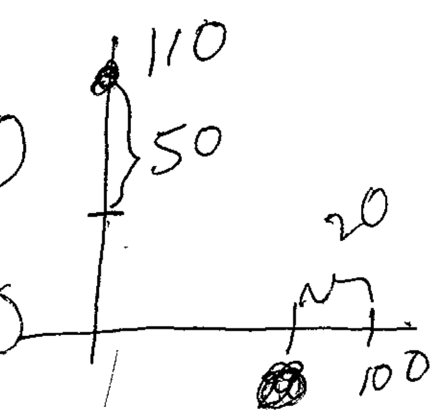
$$\frac{dr}{dt} = \frac{x\cdot x' + y y'}{r}$$

$x' = -20\text{mph}$ $\quad y' = -50\text{mph}$

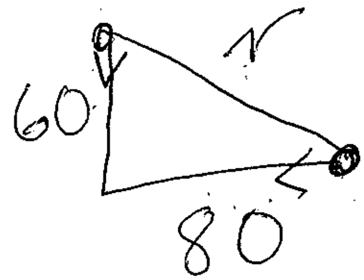
1a) When $t=1$

$y(1) = 110 - 50(1) = 60$

$x(1) = 100 - 20(1) = 80$



$r(1) = ?$



$r^2 = 60^2 + 80^2$

$\Rightarrow r(1) = 100$

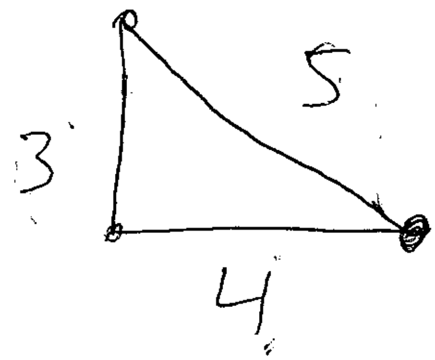
$$r' = \frac{xx' + yy'}{r} = \frac{(8\cancel{0})(-2\cancel{0}) + 6\cancel{0}(-5\cancel{0})}{10\cancel{0}}$$

$$= -16 - 30 = -46 \text{ mph}$$

negative sign means distance decreasing

The two cars get closer approachin each other at a rate at 46 mph

SIDE NOTE



$$3^2 + 4^2 = 5^2$$

Hence ~~a~~ triangles similar to the above ~~a~~ often appear exams

b) $t = 3$

$$x(3) = 100 - 3(20) = 40$$

$$y(3) = 110 - 3(50) = -40$$

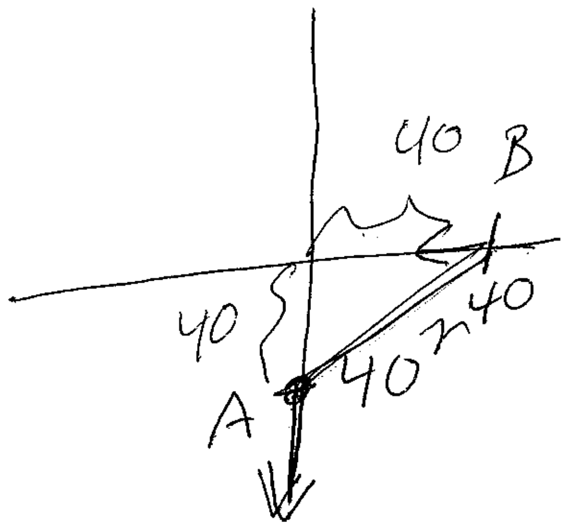
$$r(3) = 40\sqrt{2}$$

$$\boxed{\begin{array}{l}\text{velocities constant} \\ x' = -20 \text{ mph} \\ y' = -50 \text{ mph}\end{array}}$$

$$40^2 + 40^2 = r^2$$

$$2(40)^2 = r^2$$

$$r = 40\sqrt{2}$$



$$r' = \frac{-20x - 50y}{r}$$

$$= \frac{-20(\cancel{40}) - 50(\cancel{-40})}{\cancel{40}\sqrt{2}} = \boxed{\frac{30}{\sqrt{2}} \text{ mph}}$$

$\frac{30}{\sqrt{2}} > 0 \Rightarrow$ cars moving apart, getting further away from each other

Suppose car A is 110 miles north of an intersection and traveling south at 50 mph. Suppose car B is 100 miles east of the same intersection and traveling west at 20 mph. 1a.) At what rate are the cars approaching each other after 1 hr? 1b.) After 3 hrs

1.) What do you need to find: $\frac{dr}{dt}$

2.) What formula can you use which involves the variable of interest (in this case r)?

$$\boxed{x^2 + y^2 = r^2}$$

3.) What do you know about the variables in the above equation? In particular are any of the constant?

Note whether or not a variable represents a constant is very important when taking the derivative.

In this case, x, y, r are NOT constants. We do know that $x(0) = 100$, $y(0) = 110$, $x'(t) = -20$, and $y'(t) = -50$. Note the sign of the derivative indicates direction. Make sure you use the correct sign. In this case we decided that the positive x direction is east and the positive y direction represents north.

4.) Take derivative of both sides w.r.t. variable t of $x^2 + y^2 = r^2$

$$\boxed{\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = \cancel{2}r\frac{dr}{dt}}.$$

$$x(t)\frac{dx}{dt} + y(t)\frac{dy}{dt} = r(t)\frac{dr}{dt}$$

In this case $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are constant w.r.t time, but $\frac{dr}{dt}$ is a function of t which is not constant.

$\frac{dx}{dt} = -20$, and $\frac{dy}{dt} = -50$

$$\text{Hence } -20x(t) - 50y(t) = r(t)\frac{dr}{dt}$$

a.) At what rate are cars approaching each other after 1 hour?

Note $x(1) = 100 - 20 = 80$ and $y(1) = 110 - 50 = 60$.

Use $x^2 + y^2 = r^2$ to find $r(1)$: $r(1) = \sqrt{80^2 + 60^2}$
$$= \sqrt{10^2(8^2 + 6^2)} = 10\sqrt{64 + 36} = 10\sqrt{100} = 100.$$

Hence when $t = 1$: $100\frac{dr}{dt} = -20(80) - 50(60)$

Thus $\frac{dr}{dt} = -2(8) - 5(6) = -16 - 30 = -46$

Note answers to word problems must use words (including units):
After 1 hour, the cars approach each other at the rate of 46 mph

Note the symbol for the negative sign is not included in the answer. Instead we use it to determine that the cars are getting closer instead of moving apart.

b.) After 3 hours

$x(3) = 100 - 3(20) = 40$ and $y(3) = 110 - 3(50) = -40$.

The negative sign indicates that car A has crossed the intersection and hence is 40 miles south of the intersection when $t = 3$.

Use $x^2 + y^2 = r^2$ to find $r(3)$:
$$r(3) = \sqrt{40^2 + (-40)^2} = \sqrt{40^2(2)} = 40\sqrt{2}.$$

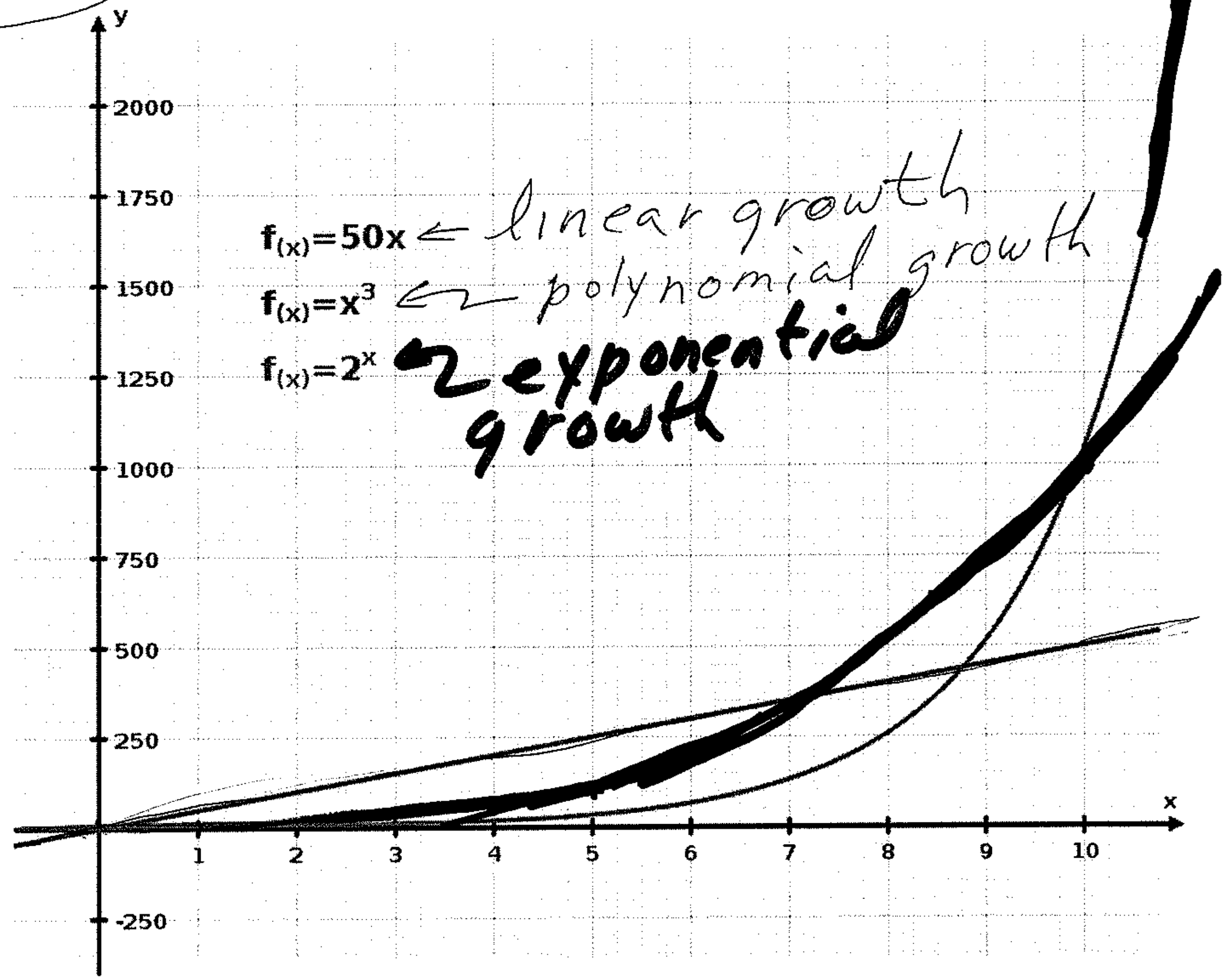
Hence when $t = 3$: $40\sqrt{2}\frac{dr}{dt} = -20(40) - 50(-40)$

Thus $\frac{dr}{dt} = \frac{-20+50}{\sqrt{2}} = \frac{30}{\sqrt{2}}$

Note ~~answers to word problems must use~~ words (including units):
After 3 hour, the cars moving apart at the rate of $\frac{30}{\sqrt{2}}$ mph

Since $\frac{dr}{dt}$ is positive, we know cars are moving apart instead of getting closer.

4.1



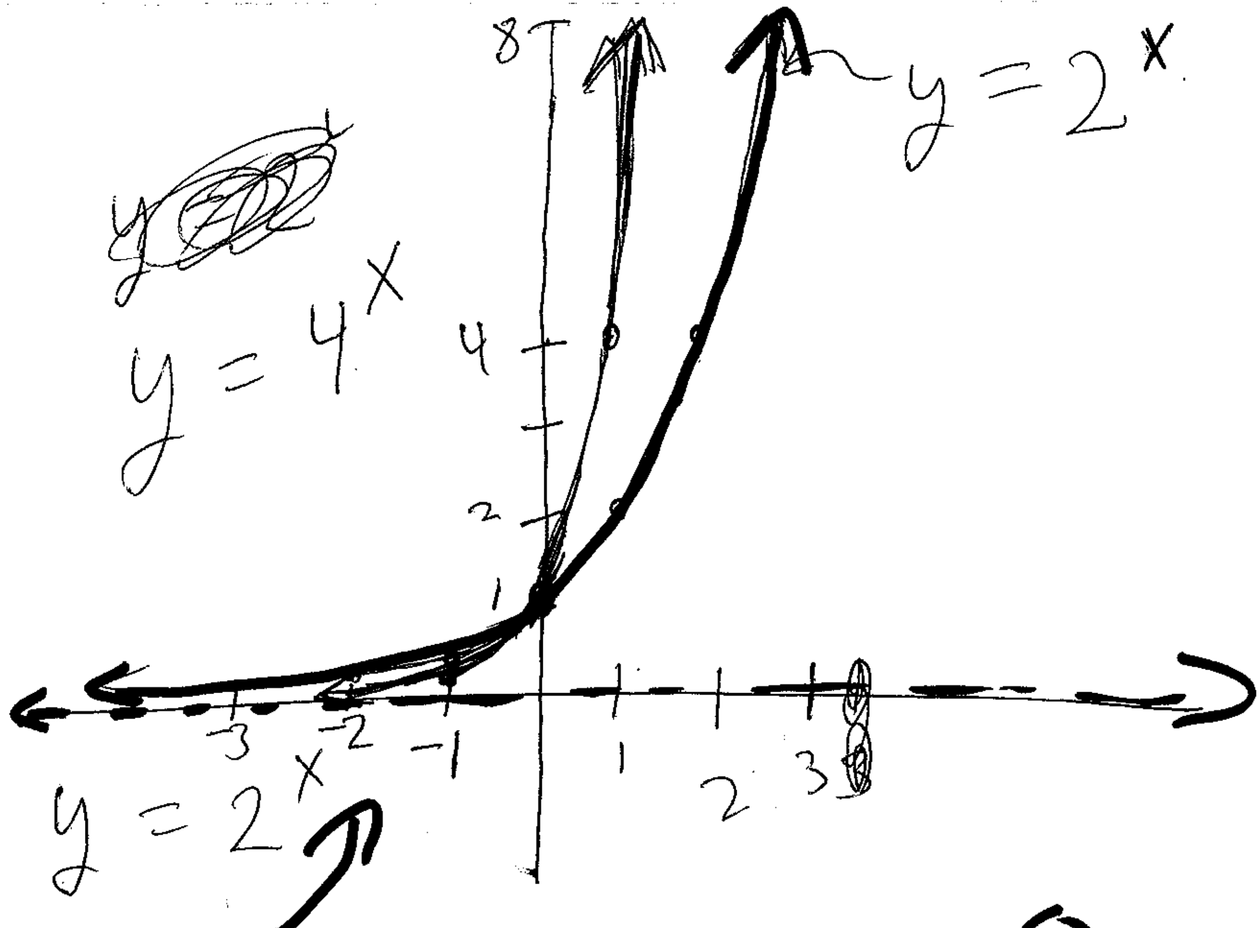
$2^{20} = 1,048,576$

$(1.9)^{20} = 375,900.$

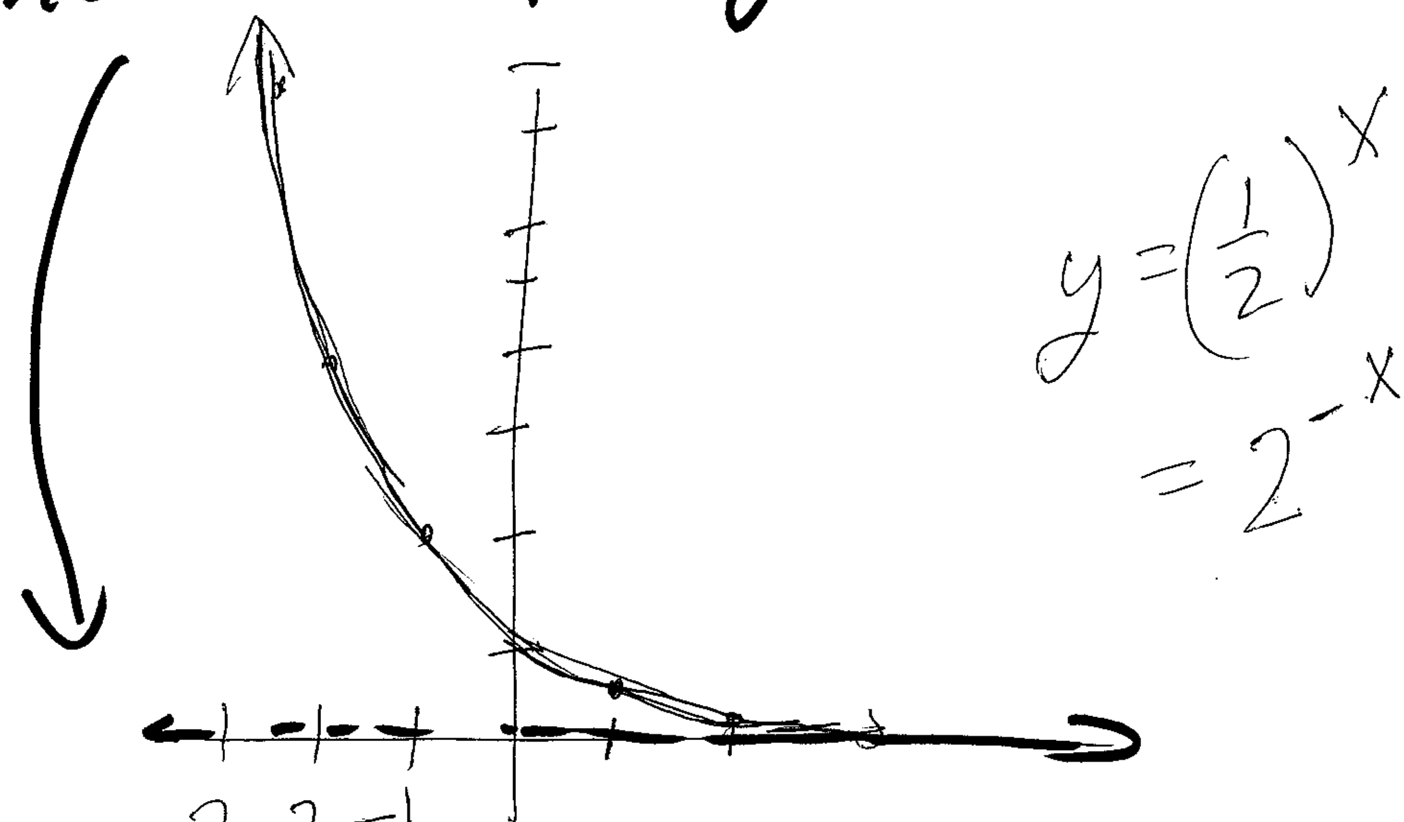
$(1.9)^{10} = 613.107$

For large x

$50x < x^3 << 2^x$



horiz asy: $y = 0$



$$f(x) = e^x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

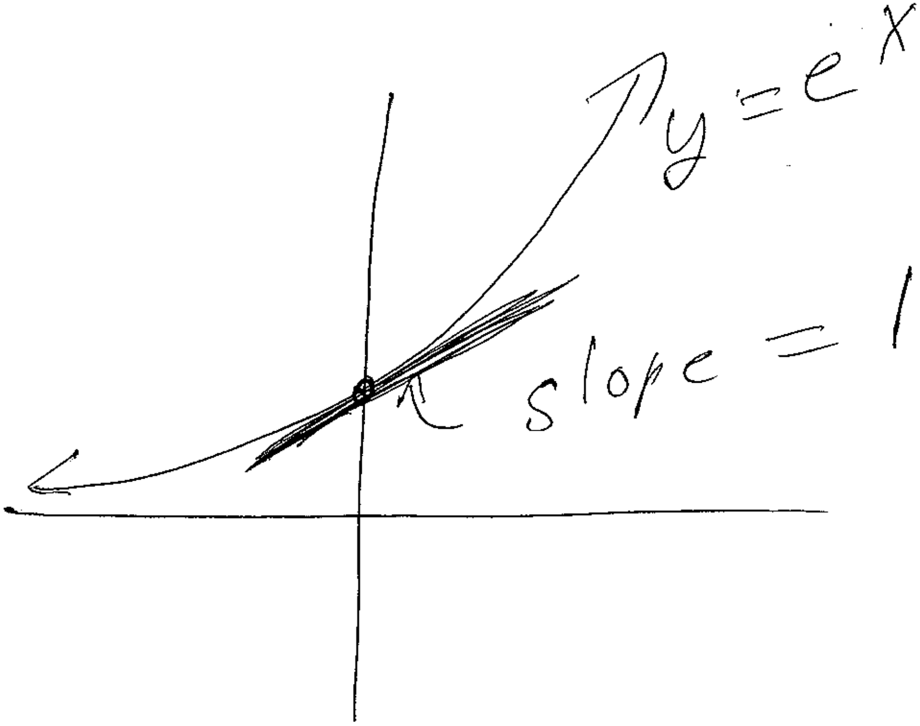
$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}$$

$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^x \lim_{h \to 0} \frac{e^{0+h} - e^0}{0+h-0}$$

$$= e^x \cdot f'(0) = e^x \cdot 1 = e^x$$



$$\left[e^{x^2+3x}\right]' =$$

$$\left(e^{x^2+3x}\right)\cdot\left(x^2+3x\right)'$$

$$= \left(e^{x^2+3x}\right)\left(2x+3\right)$$

$$\left(e^{4x}\right)' = \left(e^{4x}\right) \cdot (4x)'$$
$$= 4e^{4x}$$

$$\left[e^{2x} \cdot \sin(3e^{x})\right]' =$$
$$e^{2x}\left(\cos(3e^{x})\right) \cdot (3e^{x})$$
$$+ 2e^{2x} \sin(3e^{x})$$
