

Suppose  $f : N \rightarrow M$ . Then  $d_p f : T_p(N) \rightarrow T_p(M)$ .

If  $df_p$  is 1 – 1, for all  $p \in N$ , then  $f$  is called an *immersion*.

I.e.,  $f$  is an immersion iff  $f$  has rank  $n$

If  $df_p$  is onto for all  $p \in N$ , then  $f$  is called a *submersion*.

I.e.,  $f$  is an submersion iff  $f$  has rank  $m$

Defn. Suppose  $f : M \rightarrow N$  is smooth.

$p \in M$  is a *critical point* and  $f(p)$  is a *critical value* if  $\text{rank } df_p < n$ .

If  $p \in M$  is not a critical point, then it is a *regular point*.

If  $q \in N$  is not a *critical value*, then it is a *regular value*.

Note:  $q \in N$  is a regular value iff  $f^{-1}(q) = \emptyset$  or  $\forall p \in f^{-1}(q), df_p = n$ .

Defn:  $K$  is a *m-submanifold* of  $N$  if  $\forall q \in K \subset N, \exists g^{\text{smooth}} : V^{\text{open}} \subset N \rightarrow \mathbf{R}^{n-m}, q \in V$  such that

1.)  $g$  is smooth

2.)  $K \cap V = g^{-1}(0)$  and

3.)  $\text{rank } d_p g = n - m$

Defn: Suppose  $f : M \rightarrow N$  is a 1 – 1 immersion, and suppose  $f : M \rightarrow f(M)$  is a homeomorphism, where  $f(M) \subset N$  has the relative topology. Then  $f$  is an *embedding*, and  $f(M)$  is an embedded submanifold.

Thm.  $f : M \rightarrow N$  embedding implies  $f(M)$  is a submanifold of  $N$ .

Thm 2.3.13: Let  $q$  be a regular value of  $f : M \rightarrow N$ . Then either  $f^{-1}(q) = \emptyset$  or  $f^{-1}(q)$  is an  $(m - n)$ -submanifold of  $M$ .