Randell's Submanifolds (2.3) = Boothby's Regular submanifold (III.5):

If $K$ is a submanifold of $M$, then $K$ has the subspace topology.
If $(\phi_\alpha, U_\alpha)$ is a chart for $M$ such that $\phi_\alpha(U_\alpha) = \Pi_1^m(-\epsilon, \epsilon)$ and $\phi_\alpha(U_\alpha \cap K) = \Pi_1^k(-\epsilon, \epsilon) \times \Pi_{k+1}^m\{0\}$ , then $(\phi_\alpha|_{U_\alpha \cap K}, U_\alpha \cap K)$, $\phi_\alpha|_{U_\alpha \cap K} : U_\alpha \cap K \to \Pi_1^k(-\epsilon, \epsilon)$ is a chart for $K$.

Prop: If $U^{open} \subset M^m$, then $U$ is an $m$-dimensional submanifold of $M$.

Prop: If $K$ is a submanifold of $M$, then $i : K \to M$, $i(k) = k$, the inclusion map is smooth.

Ex: Find a counterexample to the above if we replace the hypothesis $K$ is a submanifold of $M$ with $K \subset M$.

Prop: If $f : N \to M$ is smooth and if $H$ is a submanifold of $N$, then $f : H \to M$ is smooth

Ex: Find a counterexample to the above if we replace the hypothesis $H$ is a submanifold of $N$ with $H \subset N$.

Prop: If $f : N \to M$ is smooth and if $K$ is a submanifold of $M$ and if $f(N) \subset K$, then $f : N \to K$ is smooth.

Ex: Find a counterexample to the above if we replace the hypothesis $K$ is a submanifold of $M$ with $K \subset M$.

Boothy III.6 = Randell Chapter 1.3

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Defn: $G$ is a *topological group* if

1.) $(G, *)$ is a group

2.) $G$ is a topological space.

3.) $* : G \times G \to G$, $*(g_1, g_2) = g_1 * g_2$, and
$In : G \to G$, $In(g) = g^{-1}$ are both continuous functions.

Defn: $G$ is a *Lie group* if

1.) $G$ is a group

2.) $G$ is a smooth manifold.

3.) $*$ and $In$ are smooth functions.

Ex: $Gl(n, \mathbf{R})$ = set of all invertible $n \times n$ matrices is a Lie group:

1.) $(Gl(n, \mathbf{R}), matrix\ multiplication)$ is a group

2.) $(Gl(n, \mathbf{R})$ is a smooth manifold.

3.) $*(Gl(n, \mathbf{R}) \times (Gl(n, \mathbf{R}) \to (Gl(n, \mathbf{R}),$

$$*(A, B) = AB \text{ and}$$

$In : (Gl(n, \mathbf{R}) \to (Gl(n, \mathbf{R})$

$In(A) = A^{-1}$ are smooth functions.

Ex: $(\mathbf{C} - \{\mathbf{0}\}, \cdot)$, is a Lie group.

Thm: If $G$ is a Lie group and $H$ is a submanifold, then $H$ is a Lie group.

Ex: $(S^1, \cdot)$

Ex: $G_1, G_2$ lie groups implies $G_1 \times G_2$ is a lie group.

Ex: $T^n = S^1 \times ... \times S^1$ is a Lie group.

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The following maps are diffeomorphisms:

$In : G \to G$, $In(g) = g^{-1}$.

For $a \in G$,

$L_a : G \to G$, $L_a(g) = ag$

$R_a : G \to G$, $R_a(g) = ga$

Ex: $O(n) = \{M \in GL(n, \mathbf{R}) \mid M^t M = I\}$ is a Lie group.

Ex: $Sl(n, \mathbf{R}) = \{M \in GL(n, \mathbf{R}) \mid det(M) = 1\}$ is a Lie group.

Defn: $F$ is a *homomorphism* of Lie groups if $F$ is an algebraic homomorphism of Lie groups and $F$ is smooth.

Ex: $F : GL(n, \mathbf{R}) \to \mathbf{R} - \{\mathbf{0}\}$, $F(M) = det(M)$ is a homomorphism.

Randell's Submanifolds (2.3) = Boothby's Regular submanifold (III.5):

$K \subset N$ is a $k$-submanifold of $N$ if $\forall p \in K$, there exists, Suppose $f : N \to M$ is smooth and has constant rank $k$. If $q \in M$, then $f^{-1}(q)$ is a submanifold of $N$ of dimension $n - k$.

Proof: Let $p \in f^{-1}(q)$. By the rank theorem, Ex: $F : (\mathbf{R}, +) \to (S^1, \cdot)$, $F(t) = e^{2\pi i t}$ is a homomorphism.

Ex: $F : (\mathbf{R}^n, +) \to (T^n, \cdot)$, $F(t_1, ..., t_n) = (e^{2\pi i t_1}, ..., e^{2\pi i t_n}$ is a homomorphism.

Thm: If $F : G_1 \to G_2$ is a homomorphism of Lie groups, then

1.) rank(F) is constant.

2.) kernel of $F = F^{-1}(e)$ is a closed submanifold

3.) $F^{-1}(e)$ is a Lie group.

4.) $dim(ker\ F) = dim(G_1) - rank(F)$

Thm: If $H$ is a submanifold and an algebraic subgroup of $G$, then $H$ is closed in $G$.

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Defn: $G$ = group, $X$ = set. $G$ *acts on* $X$ (on the left) if $\exists \sigma : G \times X \to X$ such that

1.) $\sigma(e, x) = x \quad \forall x \in X$

2.) $\sigma(g_1, \sigma(g_2, x)) = \sigma(g_1 g_2, x)$

Notation: $\sigma(g, x) = gx$.
Thus 1) $ex = x$; 2) $g_1(g_2 x) = (g_1 g_2)(x)$.

If $G$ is a Lie group and $X$ is a smooth manifold, then we require $\sigma$ to be smooth.

Defn: The *orbit* of $x \in X =$
$G(x) = \{y \in X \mid \exists g \text{ such that } y = gx\}$

Note:
1.) $x \in G(x)$  2.) If $G(x) \cap G(y) \neq \emptyset$, then $G(x) = G(y)$

Thus we can use an action of $G$ to partition $X$ into disjoint subsets.

Defn: If $G$ acts on $X$, then $X/G = X/\sim$ where $x \sim y$ iff $y \in G(x)$ iff $\exists g$ such that $y = gx$.

If $X$ is a topological space, then $X/G = X/\sim$ is a topological space with the quotient topology.

When is $X/G = X/\sim$ a manifold?

Ex: $G = (\mathbf{Z}, +)$, $M = \mathbf{R}$, $\sigma(n, x) = n + x$.

$M/G =$

Ex: $G = (\mathbf{Z} \times \mathbf{Z}, +)$, $M = \mathbf{R}^2$,
$\sigma((n, m), (x, y)) = (n + x, m + y)$.

$M/G =$

Ex: $G = (\mathbf{Z_2}, +)$, $M = S^n$, $\sigma(0, x) = x$, $\sigma(1, x) = -x$, .

$M/G =$