

Randell A1:

Defn: Suppose $f : M \rightarrow N$ where M and N are smooth manifolds. f is *smooth* if for all $p \in M$, \exists charts (ϕ, U) and (φ, V) and such that $p \in U$, $f(p) \in V$, $f(U) \subset V$ and $\varphi \circ f \circ \phi^{-1}$ is smooth.

Note this definition is equivalent to:

Suppose $f : M \rightarrow N$ where M and N are smooth manifolds. f is *smooth* if for all $p \in M$ and for all charts (ϕ, U) and (φ, V) such that $p \in U$, $f(p) \in V$, $f(U) \subset V$, then $\varphi \circ f \circ \phi^{-1}$ is smooth.

Suppose $g : M \rightarrow W$ and $f : W \rightarrow N$ are smooth. Let M be an m -manifold, W a k -manifold, and N an n -manifold.

Claim $f \circ g : M \rightarrow N$ is smooth.

Let $p \in M$. g smooth implies \exists charts (ϕ_1, U_1) and (φ_1, V_1) such that $p \in U_1$, $\phi_1(p) \in V_1$, $g(U_1) \subset V_1$ and $\varphi_1 \circ g \circ \phi_1^{-1} : \phi(U_1) \subset \mathbb{R}^m \rightarrow \varphi_1(V_1) \subset \mathbb{R}^k$ is smooth.

Let (φ_2, V_2) be a chart such that $f(g(p)) \in V_2$. Let $V_3 = f|_W^{-1}(V_2) \cap V_1$. Then $g(p) \in V_3$ and $f(V_3) \subset V_2$

f smooth implies f is continuous. Thus V_3 is open in W and $(\varphi_1|_{V_3}, V_3)$ is a chart.

f smooth implies $\varphi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(V_3) \subset \mathbb{R}^k \rightarrow \varphi_2(V_2) \subset \mathbb{R}^n$ is smooth.

Thus $(\varphi_2 \circ f \circ \varphi_1^{-1}) \circ (\varphi_1 \circ g \circ \phi_1^{-1}) = \varphi_2 \circ f \circ g \circ \phi_1^{-1} : \phi(U_1) \subset \mathbb{R}^m \rightarrow \varphi_2(V_2) \subset \mathbb{R}^n$ is smooth.

Thus $f \circ g$ is smooth

A2

$T_p(M) = \{v : G(p) \rightarrow \mathbf{R} \mid v \text{ is linear and satisfies the Leibniz rule} \}$

Given a chart (U, ϕ) at p where $\phi(p) = \mathbf{0}$, the *standard basis* for $T_p(M) = \{v_1, \dots, v_m\}$, where $v_i = D_{\alpha_i}$ and for some $\epsilon > 0$, $\alpha_i : (-\epsilon, \epsilon) \rightarrow M$, $\alpha_i(t) = \phi^{-1}(0, \dots, t, \dots, 0)$

B1) Let $U = \Gamma(f)$, $\phi : \Gamma(f) \rightarrow \mathbf{R}^2$, $\phi(x, y, z) = (x, y)$.

Since the domain of f is \mathbf{R}^2 , $\Gamma(f)$ is onto.

$\Gamma(f) \subset \Gamma(f)$, $\Gamma(f)$ is open in $\Gamma(f)$, and \mathbf{R}^2 is open in \mathbf{R}^2 , thus if ϕ is a homeomorphism, $\{(\phi, \Gamma(f))\}$ is a pre-atlas.

Suppose $\phi(x_1, y_1, z_1) = \phi(x_2, y_2, z_2)$. Then $(x_1, y_1) = \phi(x_1, y_1, z_1) = \phi(x_2, y_2, z_2) = (x_2, y_2)$. Also $z_1 = f(x_1, y_1) = f(x_2, y_2) = z_2$. Thus $x_1 = x_2, y_1 = y_2, z_1 = z_2$. Hence ϕ is 1:1.

Let $\pi_{xy} : \mathbf{R}^3 \rightarrow \mathbf{R}^2$, $\pi_{xy}(x, y, z) = (x, y)$. If U is open in \mathbf{R}^2 , $\pi_{xy}^{-1}(U) = U \times \mathbf{R}$ which is open in \mathbf{R}^3 . Thus π_{xy} is continuous and $\phi = \pi_{xy}|_{\Gamma(f)}$ is continuous.

Let V be open in \mathbf{R}^2 . Then $\pi_{xy}|_{\Gamma(f)}(V) = (V \times \mathbf{R}) \cap \Gamma(f)$ is open in $\Gamma(f)$.

Thus $\phi : \Gamma(f) \rightarrow \mathbf{R}^2$ is a homeomorphism.

Since $\Gamma(f)$ has a pre-atlas, $\Gamma(f)$ is a smooth manifold.