12. Topological Spaces

Defn: A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

a.) $\emptyset, X \in \mathcal{T}$.

b.) $U_\alpha \in \mathcal{T}$ implies $\cup U_\alpha \in \mathcal{T}$

c.) $U_i \in \mathcal{T}$ implies $\cap_{i=1}^n U_i \in \mathcal{T}$

Defn: U is open if $U \in \mathcal{T}$

Ex 2a: The discrete topology on $X = \mathcal{P}(X) =$ set of all subsets of X.

Ex 2b: The indiscrete or trivial topology on $X = \{\emptyset, X\}$.

Ex 3: The finite complement topology on $X = \mathcal{T}_f$ $= \{U \mid X - U$ is finite or $X - U = X\}$.

Ex 4: The countable complement topology on X $=\mathcal{T}_c = \{U \mid X - U$ is countable or $X - U = X\}$.

Defn: Suppose the $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X such that $\mathcal{T} \subset \mathcal{T}'$. Then $\mathcal{T}'$ is **finer** or **larger** than $\mathcal{T}$ and $\mathcal{T}$ is **coarser** or **smaller** than $\mathcal{T}'$. If $\mathcal{T}'$ properly contains $\mathcal{T}$, then $\mathcal{T}'$ is **strictly finer** than $\mathcal{T}$ and $\mathcal{T}$ is **strictly coarser** than $\mathcal{T}'$.

Defn: $\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if either $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$.

13: Basis for a Topology

Defn: If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) such that

(1) For each $x \in X$, there is at least one basis element B containing x.

(2) If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

The topology $\mathcal{T}$ generated by a basis $\mathcal{B}$ is defined as follows: U is open if and only if for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$

Example 1a: The set of all open intervals in R is a basis for a topology on R (the standard topology).

Example 1b: The set of all open circular regions in R^2 is a basis for a topology on R^2 (the standard topology).

Example 2: The set of all open rectangular regions in R^2 is a basis for a topology on R^2 (the standard topology).

Note the basis in Example 1b and the basis in Example 2 both generated the same topology.

Example 3: $\{x \mid x \in X\}$ is a basis for the discrete topology on X.

Lemma 13.1: Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T} =$ set of all unions of elements of $\mathcal{B}$.

Lemma 13.2: Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology on X.

Lemma 13.3: Let $\mathcal{B}$ and $\mathcal{B}'$ be a basis for $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:

(1) $\mathcal{T}'$ is finer than $\mathcal{T}$.

(2) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Defn:
1.) $\mathcal{B} = \{(a, b) \mid a, b \in R, a < b\}$ is a basis for the standard topology on R.

2.) $\mathcal{B}' = \{[a, b) \mid a, b \in R, a < b\}$ is a basis for the lower limit topology on R. When R has this topology, we denote it by R_l.

3.) Let $K = \{\frac{1}{n} \mid n \in Z_+\}$.
$\mathcal{B}'' = \mathcal{B} \cup \{(a, b) - K \mid a, b \in R, a < b\}$ is a basis for the K-topology on R. When R has this topology, we denote it by R_K.

Lemma 13.4: The topologies R_l and R_K are strictly finer than the standard topology, but they are not comparable with one another.

Definition: A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The **topology generated by the subbasis** $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

Lemma: If $\mathcal{S}$ is a subbasis for a topology on X, then $\mathcal{B}$ = the set of all finite intersections of elements of $\mathcal{S}$ is a basis for this topology.

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14: The Order topology

(p. 24) A relation $<$ on a set A is called an **order relation** (or a **simple order** or **linear order**) if it has the following properties:

(1) (Comparability) For every $x, y \in A$ for which $x \neq y$, either $x < y$ or $y < x$.

(2) (Nonreflexivity) For no $x \in A$ does the relation $x < x$ hold.

(3) (Transitivity) If $x < y$ and $y < z$, then $x < z$.

Defn: Let X be a set with a simple order relation. Assume that X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

(1) All open intervals (a, b) in X.

(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.

(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.

The collection $\mathcal{B}$ is a basis for a topology on X which is called the **order topology**.

Note: If X has no smallest element, there are no sets of type (2). If X has no largest element, there are no sets of type (3).

Ex. 0: The order topology on $(0, 1) \cup \{5\}$

Ex. 1: The order topology on R is the standard topology on R.

Ex. 2: $R \times R$ in the dictionary order.

Ex. 3: Order topology on $Z_+ =$ discrete topology.

Ex. 4: The order topology on $X = \{1, 2\} \times Z_+$ is NOT the discrete topology.