1. FIRST CONCEPTS

The geometry which we will study consists of a set S, called *space*. The elements of the set are called *points*. Furthermore S has certain distinguished subsets called *lines* and *planes*. A little later we will introduce other special types of subsets of S, for example, *circles*, *triangles*, *spheres*, etc.

On the one hand, we want to picture these various types of subsets according to our usual conceptions of them: Lines, planes, and so forth are idealizations of objects known from experience of the physical world. For example, a line is an idealization of a piece of string streched taughtly between two points. But it is supposed to extend indefinitely in both directions, and, of course, we do not have any direct physical experience with anything of indefinite extent. Similarly a plane is supposed to be a flat surface, like a table-top, but also is supposed to extend indefinitely in all directions. (Sort of like Nebraska, but larger.) Again, we don't have any direct physical experience with flat surfaces of indefinite extent.

On the other hand, we want to try to be very careful not to use in any proof any assumptions about these objects except those which we have made explicit. Only in this way can we be sure that our arguments are actually correct, and that our conclusions are actually correct.

We will allow ourselves the use of the real numbers, and all of their usual properties.

Axiom I-1 Given two *distinct* points, there is exactly one line containing them.

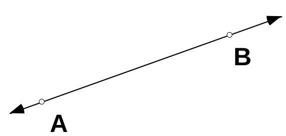

FIGURE 1.1. Axiom I-1

Remember, a line is a set of points, and containment here means containment as elements. We denote by $\overleftrightarrow{PQ}$ the line containing distinct points P and Q

We call any collection of points which lie on one line colinear and any collection of points which lie on one plane coplanar

Axiom I-2 Given three *non-colinear* points, there is exactly one plane containing them.

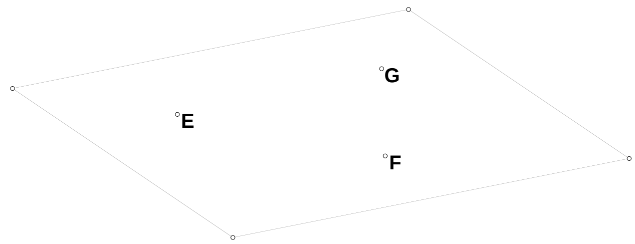

FIGURE 1.2. Axiom I-2

Axiom I-3 If two distinct points lie in a plane P, then the line containing them is a subset of P.

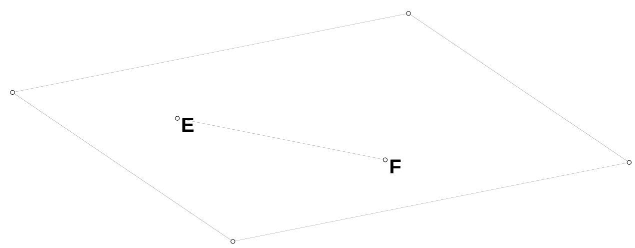

FIGURE 1.3. Axiom I-3

Axiom I-4 If two planes intersect, then their intersection is a line.

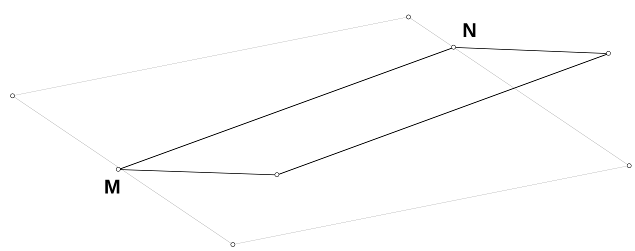

FIGURE 1.4. Axiom I-4

Theorem 1.1. *If two distinct lines intersect, then their intersection consists of exactly one point.*

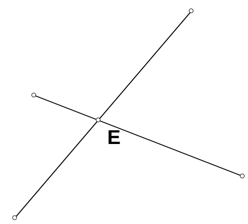

FIGURE 1.5. Theorem 1.1

Proof. We could rephrase the statement thus: if two lines are distinct, then their intersection does not contain two distinct points. The contrapositive is: If two lines L and M contain two distinct points in their intersection, then $L = M$. We prove this contrapositive statement.

Suppose L and M are lines (possibly the same, possibly distinct), and P and Q are two different points in their intersection. Since P, Q are elements of L, it follows from Axiom I-1 that $L = \overleftrightarrow{PQ}$. Likewise, since P, Q are elements of M, it follows from Axiom I-1 that $M = \overleftrightarrow{PQ}$. But then $M = \overleftrightarrow{PQ} = L$ □

Theorem 1.2. *If a line L intersects a plane $\mathcal{P}$ and L is not a subset of $\mathcal{P}$ then the intersection of L and $\mathcal{P}$ consists of exactly one point.*

Proof. Exercise, or in class. □

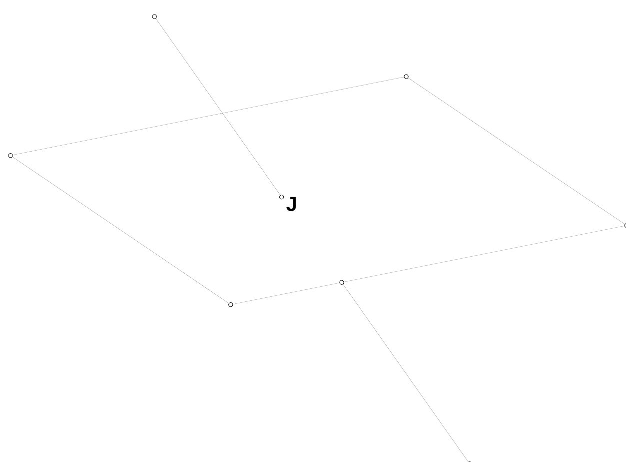

FIGURE 1.6. Theorem 1.2

So far, all the axioms (and two theorems) would be valid for a geometry with only one point P with $\{P\}$ begin both a line and an plane! So clearly the axioms so far do not force us to be talking about the geometry which we expect to talk about! Very shortly, I will give axioms which ensure that space has lots of points, but in the meanwhile let us at least assume the following:

Axiom I-5 Every line has at least two points. Every plane has at least 3 non-colinear points. And S has at least 4 non-coplanar points.

Theorem 1.3. *If L is a line, and P is a point not in L, then there is exactly one plane $\mathcal{P}$ containing $L \cup \{P\}$.*

Proof. Exercise, or in class. □

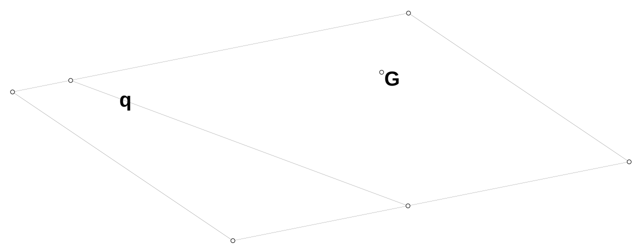

FIGURE 1.7. Theorem 1.3

Theorem 1.4. *If L and M are two distinct lines which intersect, then there is exactly one plane containing $L \cup M$.*

Proof. Exercise, or in class. □

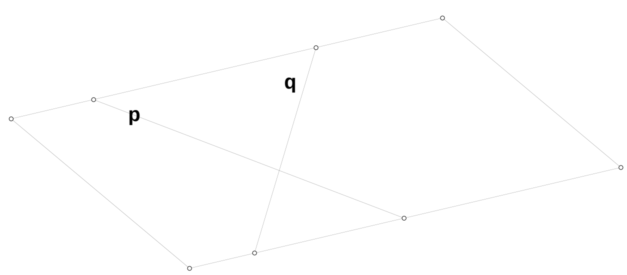

FIGURE 1.8. Theorem 1.4

2. DISTANCE

A familiar notion in geometry is that of distance. The distance between two points is the length of the line segment connecting them. In order to get things into logical order, we will actually introduce the notion of distance first, and use it to establish the notion of line segment!

Axiom D-1 **For every pair of points A, B there is a number $d(A, B)$, called the *distance from* A *to* B. Distance satisfies the following properties:**

1. $d(A, B) = d(B, A)$
2. $d(A, B) \geq 0$**, and** $d(A, B) = 0$ **if, and only if,** $A = B$**.**

Definition 2.1. A *coordinate function* on a line L is a bijective (one-to-one and onto) function f from L to the real numbers $\mathbb{R}$ which satisfies $|f(A) - f(B)| = d(A, B)$ for all $A, B \in L$. Given a coordinate function f, the number $f(A)$ is called the coordinate of the point $A \in L$.

Axiom D-2 **Every line has at least one coordinate function.**

It follows immediately that every line contains infinitely many points, because $\mathbb{R}$ is an infinite set, and a coordinate function is a one-to-one correspondence of the line with $\mathbb{R}$. Any coordinate function makes a line into a "number line" or "ruler".

Lemma 2.2. *If L is a line and $f : L \to \mathbb{R}$ is a coordinate function, then $g(A) = -f(A)$ is also a coordinate function.*

Proof. In class, or exercise. □

Lemma 2.3. *If L is a line and $f : L \to \mathbb{R}$ is a coordinate function, then for any real number s, $h(A) = f(A) + s$ is also a coordinate function.*

Proof. In class, or exercise. □

Lemma 2.4. *Let L be a line and A and B distinct points on the line L. Then there is a coordinate function f on L satisfying $f(A) = 0$ and $f(B) > 0$. Furthermore, if g is any coordinate function on L then f can be taken to have the form*

$$f(P) = \pm g(P) + s$$

for some $s \in \mathbb{R}$.

Proof. By Axiom D-2, L has a coordinate function g. Let $s = g(A)$, and define $f_1(P) = g(P) - s$. By Lemma 2.3, f_1 is also a coordinate function, and $f_1(A) = g(A) - s = 0$. Now $|f_1(B)| = |f_1(B) - f_1(A)| = d(A, B) > 0$, since $A \neq B$. If $f_1(B) > 0$, we take $f(P) = f_1(P)$. Otherwise, we take $f(P) = -f_1(P)$, which is also a coordinate function by Lemma 2.2. □

Theorem 2.5. *Let L be a line and A and B distinct points on the line L. There is exactly one coordinate function f on L satisfying $f(A) = 0$ and $f(B) > 0$.*

Proof. The previous lemma says that there is at least one such function. We have to show that there is only one. So let f, g be two coordinate functions on L satisfying $f(A) = g(A) = 0$ and $f(B) > 0, g(B) > 0$. We have to show that $f(C) = g(C)$ for all $C \in L$. In any case, we have $|f(C)| = |f(C) - f(A)| = d(A, B) = |g(C) - g(A)| = |g(C)|$. So in case $f(C)$ and $g(C)$ are both non-negative or both non-positive, they are equal. In particular, $f(B) = g(B) = d(A, B)$.

If $f(C), g(C)$ satisfy $f(C) \leq f(B)$ and $g(C) \leq g(B)$, then $g(B) - g(C) = d(B, C) = f(B) - f(C)$. Therefore, $f(C) - g(C) = f(B) - g(B) = 0$, or $f(C) = g(C)$.

The only remaining case to consider is that for some $C \in L$, one of $f(C), g(C)$ is negative and one is greater than $f(B) = g(B)$. Without loss of generality, assume $g(C) < 0$ and $g(B) < f(C)$. Then we have

$$\begin{aligned} d(C, B) &= g(B) - g(C) \\ &= (g(B) - g(A)) + (g(A) - g(C)) \\ &= d(A, B) + d(A, C), \end{aligned}$$

since $g(C) < g(A) < g(B)$. Using the coordinate function f instead, we have

$$\begin{aligned} d(A, C) &= f(C) - f(A) \\ &= (f(C) - f(B)) + (f(B) - f(A)) \\ &= d(B, C) + d(A, B), \end{aligned}$$

since $f(A) < f(B) < f(C)$. Adding the two displayed equations gives

$$d(C, B) + d(A, C) = d(A, B) + d(A, C) + d(B, C) + d(A, B),$$

and canceling like quantities on the two sides gives

$$0 = 2d(A, B).$$

But this is false, because $A \neq B$. This contradiction shows that the case under consideration cannot occur. So we always have $f(C) = g(C)$. □

Theorem 2.6. *Let f, g be two coordinate functions on a line L. Then*

$$f(P) = \pm g(P) + s,$$

for some $s \in \mathbb{R}$.

Proof. Let $A = f^{-1}(0)$, so $f(A) = 0$. Furthermore, let $B = f^{-1}(1)$, so $f(B) = 1$. According to Lemma 2.4, there is a coordinate function h of the form $h(P) = \pm g(P) + s$ which satisfies $h(A) = 0$ and $h(B) > 0$. But according to Theorem 2.5, $h = f$, so f has the desired form. □

3. Betweenness, segments, and rays

Definition 3.1. Let x, y, and z be three different real numbers. We say that y is *between* x and z if $x < y < z$ or $z < y < x$. We denote this relation by $x — y — z$

Note that $x — y — z$ is equivalent to $z — y — x$.

Lemma 3.2. *Let x, y, and z be three different real numbers. Let s be a real number. The following are equivalent:*

(a) $x — y — z$.
(b) $(x+s) — (y+s) — (z+s)$.
(c) $(-x) — (-y) — (-z)$.
(d) $(-x+s) — (-y+s) — (-z+s)$.

Proof. This is true because addition of a number to both sides of an inequality preserves the inequality, while multiplying both sides of an inequality by (-1) reverses the order of the inequality. □

Lemma 3.3. *Let L be a line, and let f, g be two coordinate functions on L. Let A, B, C be distinct points on L. The following are equivalent:*

(a) $f(A) — f(B) — f(C)$.
(b) $g(A) — g(B) — g(C)$.

Proof. Note that all the quantities $f(P), g(P)$ for P a point in L are real numbers. So the two conditions concern betweenness for real numbers.

According to Theorem 2.6, there is an $\epsilon \in \{\pm 1\}$ and a real number s such that for all points P on L, $f(P) = \epsilon g(P) + s$. Then according to Lemma 3.2, the two conditions (a) and (b) are equivalent. □

Definition 3.4. Let L be a line, and let A, B, C be distinct points on L. We say that B is *between* A and C if for some coordinate function f on L, one has $f(A) — f(B) — f(C)$. We denote this relation by $A — B — C$.

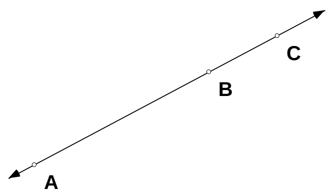

FIGURE 3.1. B is between A and C

According to Lemma 3.2, if $f(A) — f(B) — f(C)$ for *one* coordinate function f, then $f(A) — f(B) — f(C)$ for *all* coordinate functions f. So the concept of betweenness for points on a line does not depend on the choice of a coordinate function. By convention, when we assert that three points A, B, C satisfy $A — B — C$, we implicitly assert that the three points are distinct and colinear.

The next two theorems are very easy:

Theorem 3.5. *$A — B — C$ if, and only if, $C — B — A$.*

Proof. Exercise or in class. □

Theorem 3.6. *Given three distinct points on a line, exactly one of them is between the other two.*

Proof. Exercise or in class. □

Definition 3.7. Let A and B be two distinct points. The *line segment* $\overline{AB}$ is the subset of the line $\overleftrightarrow{AB}$ consisting of A, B, and the set of points C which are between A and B.

$$\overline{AB} = \{C : A — C — B\} \cup \{A, B\}$$

Theorem 3.8. *Let A, B be distinct points and let f be a coordinate system on $\overleftrightarrow{AB}$ such that $f(A) < f(B)$. Then $\overline{AB} = \{C \in \overleftrightarrow{AB} : f(A) \le f(C) \le f(B)\}$.*

Proof. Exercise or in class. □

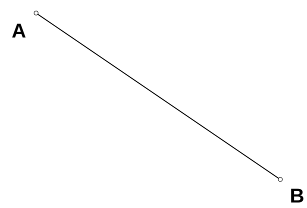

FIGURE 3.2. A Segment

Theorem 3.9. *A line segment determines its endpoints. That is, if segments* $\overline{AB}$ *and* $\overline{A'B'}$ *are equal, then* $\{A, B\} = \{A', B'\}$.

Proof. Exercise or in class. □

Definition 3.10. Let A and B be two distinct points. The *ray* $\overrightarrow{AB}$ is is the subset of the line $\overleftrightarrow{AB}$ consisting of A, B, and the set of points C such that $A — C — B$ or $A — B — C$.

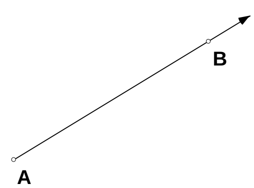

FIGURE 3.3. A Ray

Theorem 3.11. *Let* A *and* B *be two distinct points. The ray* $\overrightarrow{AB}$ *consists of those points* $C \in \overleftrightarrow{AB}$ *such that* C does not *satisfy* $C — A — B$.

Proof. Exercise or in class. □

Theorem 3.12. *Let* A *and* B *be two distinct points. Let* f *be a coordinate function on* $\overleftrightarrow{AB}$ *such that* $f(A) = 0$ *and* $f(B) > 0$. *Then The ray* $\overrightarrow{AB}$ *consists of those points* $C \in \overleftrightarrow{AB}$ *such that* $f(C) \geq 0$.

Proof. Exercise or in class. □

Theorem 3.13. *A ray determines its endpoint. That is, if rays* $\overrightarrow{AB}$ *and* $\overrightarrow{A'B'}$ *are equal, then* $A = A'$.

Proof. Exercise or in class. □

Theorem 3.14. *A ray is determined by its endpoint and any other point on the ray. That is, if* $C \in \overrightarrow{AB}$ *and* $C \neq A$*, then* $\overrightarrow{AC} = \overrightarrow{AB}$.

Proof. Exercise or in class. □

Definition 3.15. An *angle* is the union of two rays with the same endpoint, not contained in one line. The two rays are called the sides of the angle. The common endpoint is called the vertex of the angle. The angle $\overrightarrow{AB} \cup \overrightarrow{AC}$ is denoted $\angle BAC$ (or equally well $\angle CAB$.)

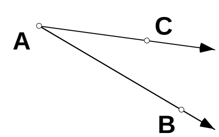

FIGURE 3.4. An Angle

Remark 3.16. The union of two *distinct* rays with a common endpoint, which *do* lie on one line, *is* the line. (Proof?) So we will sometimes call a line with a distinguished point on the line a *straight angle*.

Definition 3.17. Let A, B, C be non-colinear points. The *triangle* $\triangle ABC$ is the union of the segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. The three segments are called the *sides of the triangle*. The angles $\angle ABC$, $\angle BCA$, $\angle CAB$ are called the *angles of the triangle*. One often denotes these angles by $\angle A$, $\angle B$, and $\angle C$, respectively. One says that $\angle C$ and side $\overline{AB}$ are *opposite*, and similarly for the other angles and sides.

Theorem 3.18. *A triangle determines its vertices. That is, if* $\triangle ABC = \triangle DEF$*, then* $\{A, B, C\} = \{D, E, F\}$.

Proof. The proof of this is surprisingly tricky, and requires several steps. We will probably skip it. □

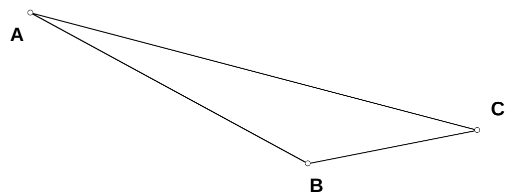

FIGURE 3.5. A Triangle

The next result is slightly technical. It gives a characterization of betweenness (and therefore of line segments).

Theorem 3.19. *Let A, B, C be distinct points on a line. The following are equivalent:*
(a) $A — B — C$.
(b) $d(A, C) = d(A, B) + d(B, C)$.

Proof. It is possible to choose a coordinate function f on L such that $f(A) < f(B)$, by Theorem 2.5.

Suppose $A — B — C$. Then $f(A) < f(B) < f(C)$, so

$$\begin{aligned} d(A, C) &= f(C) - f(A) \\ &= (f(C) - f(B)) + (f(B) - f(A)) \\ &= d(B, C) + d(A, B). \end{aligned}$$

Thus we have (a) implies (b).

Suppose now that (b) holds. According to Theorem 3.6, exactly one of the conditions is satisfied:

1. $B — A — C$.
2. $A — C — B$.
3. $A — B — C$.

Our strategy is to eliminate the first two possibilities, leaving only the third.

Suppose we have $B — A — C$. It follows that

$$d(B, C) = d(B, A) + d(A, C), \tag{3.1}$$

by the (already proved) implication (a) implies (b).

Now adding this equation and the equation in condition (b), and then canceling like terms on the two sides gives

$$0 = 2d(B, A), \tag{3.2}$$

so that $A = B$ by Axiom D-1. This contradicts our original assumptions, so it cannot be true that $B — A — C$.

The second possibility is eliminated in exactly the same way. This leaves only the third possibility, and proves the implication (b) implies (a). □

This theorem gives us a not so obvious characterization of line segments:

Corollary 3.20. *Let A and C be distinct points, and let B be a third point on $\overleftrightarrow{AC}$, possibly equal to one of A, C. The following are equivalent:*

(a) *B is on the line segment $\overline{AC}$.*

(b) $d(A, C) = d(A, B) + d(B, C)$.

Theorem 3.21. *Given two distinct points A and B on a line L, there is a point M on L such that $A — M — B$ and there is a point E on L such that $A — B — E$.*

Proof. Let f be a coordinate function on L chosen such that $0 = f(A) < f(B)$, which is possible by Theorem 2.5. Let $m = f(B)/2$ and $e = 2f(B)$. Since f is a bijection between L and $\mathbb{R}$, there exist unique points M and E on L such that $f(M) = m$ and $f(E) = e$. Now we have $f(A) < f(M) < f(B)$, and $f(A) < f(B) < f(E)$, so $A — M — B$ and $A — B — E$. □

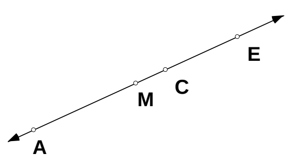

FIGURE 3.6. Theorem 3.13

Given 4 distinct points A, B, C, D on a line, we write $A — B — C — D$ if all the relations hold: $A — B — C$, $A — B — D$, $A — C — D$, and $B — C — D$.

Theorem 3.22. *Any four points on a line can be named in exactly one order A, B, C, D such that $A — B — C — D$.*

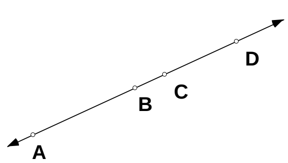

FIGURE 3.7. Theorem 3.14

Definition 3.23. The *length* of a line segment $\overline{AB}$ is $d(A, B)$. The length is sometimes denoted by $\ell(\overline{AB})$. Two segments are said to be *congruent* if they have the same length. One denotes congruence of segments by $\overline{AB} \cong \overline{CD}$.

Theorem 3.24. (Segment addition and subtraction) *Suppose A, B, C are colinear with A — B — C and A', B', C' are colinear with A' — B' — C'.*

(a) *If $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$.*
(b) *If $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$, then $\overline{BC} \cong \overline{B'C'}$.*

Proof. This is immediate from the definition of congruence and the implication (a) implies (b) in Theorem 3.19. □

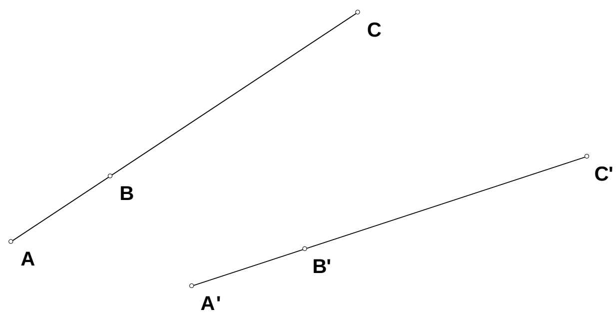

FIGURE 3.8. Theorem 3.24

4. SOME LOGIC, AND SOME PROPERTIES OF FUNCTIONS

4.1. **Quantifiers.** One frequently makes statements in mathematics which assert that all the elements in some set have a certain property, or that there exists at least one element in the set with a certain property. For example:

- For every real number x, one has $x^2 \geq 0$.
- For all lines L and M, if $L \neq M$ and $L \cap M$ is non-empty, then $L \cap M$ consists of exactly one point.
- There exists a positive real number whose square is 2.
- Let L be a line. Then there exist at least two points on L.

Statements containing one of the phrases "for every", "for all", "for each", etc. are said to have a *universal quantifier*. Such statements typically have the form:

- *For all x, $P(x)$,*

where $P(x)$ is some assertion about x. The first two examples above have universal quantifiers.

Statements containing one of the phrases "there exists," "there is," "one can find," etc. are said to have an *existential quantifier*. Such statements typically have the form:

- *There exists an x such that* $P(x)$*,*

where $P(x)$ is some assertion about x. The third and fourth examples above contain existential quantifiers.

One thing to watch out for in mathematical writing is the use of implicit universal quantifiers, which are usually coupled with implications. For example,

- If x is a non-zero real number, then x^2 is positive

actually means,

- For all real numbers x, if $x \neq 0$, then x^2 is positive,

or

- For all non-zero real numbers x, the quantity x^2 is positive.

4.2. **Negation of Quantified Statements.** Let us consider how to form the negation of sentences containing quantifiers. The negation of the assertion that every x has a certain property is that *some* x does not have this property; thus the negation of

- *For every x,* $P(x)$.

is

- *There exists an x such that not* $P(x)$.

For example the negation of the (true) statement

- For all non-zero real numbers x, the quantity x^2 is positive

is the (false) statement

- There exists a non-zero real numbers x, such that $x^2 \leq 0$.

Similarly the negation of a statement

- *There exists an x such that* $P(x)$.

is

- *For every x, not* $P(x)$.

For example, the negation of the (true) statement

- *There exists a real number x such that* $x^2 = 2$.

is the (false) statement

- *For all real numbers x,* $x^2 \neq 2$.

In order to express complex ideas, it is quite common to string together several quantifiers. For example

- *For every positive real number x, there exists a positive real number y such that $y^2 = x$.*
- *For every natural number m, there exists a natural number n such that $n > m$.*
- *For every pair of distinct points p and q, there exists exactly one line L such that L contains p and q.*

All of these are true statements.

There is a rather nice rule for negating such statements with chains of quantifiers: one runs through chain changing every universal quantifier to an existential quantifier, and every existential quantifier to a universal quantifier, and then one negates the assertion at the end.

For example, the negation of the (true) sentence

- *For every positive real number x, there exists a positive real number y such that $y^2 = x$.*

is the (false) statement

- *There exists a positive real number x such that for every positive real number y, one has $y^2 \neq x$.*

4.3. **Order of quantifiers.** It is important to realize that the order of universal and existential quantifiers cannot be changed without utterly changing the meaning of the sentence. For example, if you start with the true statement:

- *For every positive real number x, there exists a positive real number y such that $y^2 = x$*

and reverse the two quantifiers, you get the totally absurd statement:

- *There exists a positive real number x such that for every positive real number y, one has $y^2 = x$.*

4.4. **Properties of functions.** We recall the notion of a *function from A to B* and some terminology regarding functions which is standard throughout mathematics. A function f from A to B is a rule which gives for each element of $a \in A$ an "outcome" in $f(a) \in B$. A is called the *domain* of the function, B the *co-domain*, $f(a)$ is called the *value* of the function at a, and the set of all values, $\{f(a) : a \in A\}$, is called the *range* of the function.

In general, the range is only a subset of B; a function is said to be *surjective*, or *onto*, if its range is all of B; that is, for each $b \in B$, there exists an $a \in A$, such that $f(a) = b$. Figure 4.1 exhibits a surjective function. Note that the statement that a function is surjective has to be expressed by a statement with a string of quantifiers.

A function f is said to be *injective*, or *one-to-one*, if for each two distinct elements a and a' in A, one has $f(a) \neq f(a')$. Equivalently, for all $a, a' \in A$, if $f(a) = f(a')$ then $a = a'$. Figure 4.2 displays an injective and a non- injective function.

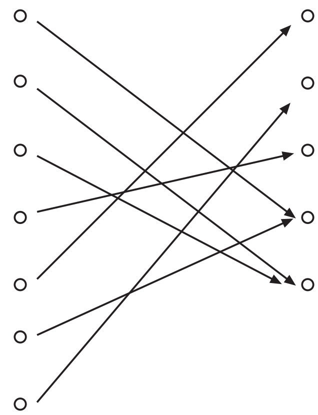

FIGURE 4.1. A Surjection

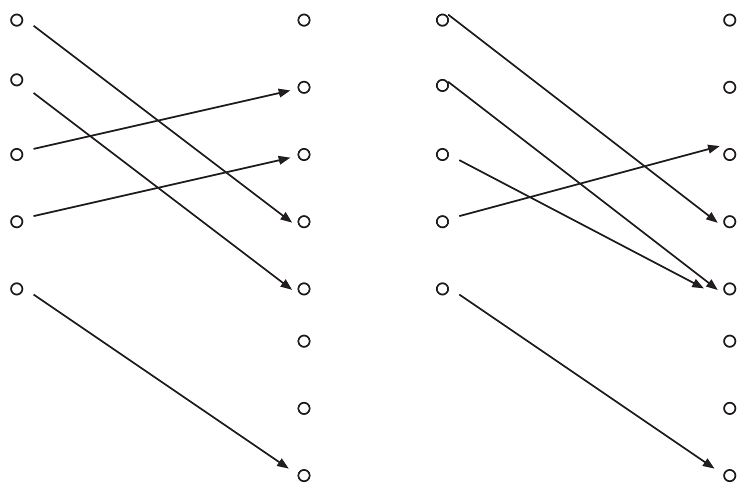

FIGURE 4.2. Injective and Non-injective functions

Finally f is said to be *bijective* if it is both injective and surjective. A bijective function (or *bijection*) is also said to be a *one-to-one correspondence* between A and B, since it matches up the elements of the two sets one-to-one. When f is bijective, there is an *inverse function* f^{-1} defined by $f^{-1}(b) = a$ if, and only if, $f(a) = b$. Figure 4.3 displays a bijective function.

If $f : X \to Y$ is a function and A is a subset of X, we write $f(A)$ for $\{f(a) : a \in A\} = \{y \in Y : \text{ there exists } a \in A \text{ such that } y = f(a)\}$. We refer to $f(A)$ as the *image of A under f*. If B is a subset of Y, we write $f^{-1}(B)$ for $\{x \in X : f(x) \in B\}$. We refer to $f^{-1}(B)$ as the *preimage of B under f*.

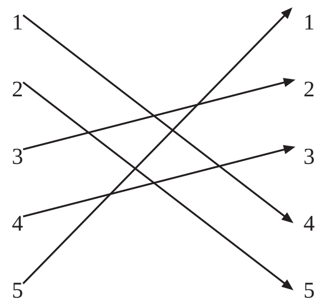

FIGURE 4.3. A Bijection

5. SEPARATION OF A PLANE BY A LINE

According to our usual conception of lines and planes, a line L contained in a plane $\mathcal{P}$ divides the plane into two "halves," one one each "side" of the line. Given two points on one side of the line, it is possible to trace a curve from one point to the other which does not cross the line L. But given two points on opposite sides of the line, any curve from one to the other will cross the line. These statements do not follow from our previous axioms, so we need to assert them as a new axiom.

First we need a definition:

Definition 5.1. A set S is *convex* if, for each two distinct points $A, B \in S$, the line segment $\overline{AB}$ is a subset of S.

For example, it follows from Axiom I-3 that a plane is a convex set.

Exercise 5.2.

1. Every line is convex.
2. Every line segment is convex.
3. Every ray is convex.
4. The set of points on a ray other than the endpoint is convex.

Exercise 5.3. Draw some pictures of convex and non-convex subsets of a plane.

Axiom PS (Plane separation axiom) Let L be a line and $\mathcal{P}$ a plane containing L. Then $\mathcal{P} \setminus L$ (the set of points on $\mathcal{P}$ which are not on L) is the union of two sets H_1 and H_2 with the properties:

1. **H_1 and H_2 are non-empty and convex.**
2. **Whenever P and Q are points such that $P \in H_1$ and $Q \in H_2$, the segment $\overline{PQ}$ intersects L.**

One calls H_1 and H_2 the two *half-planes determined by* L. One says that two points both contained in one of the half-planes *are on the same side of* L, and that

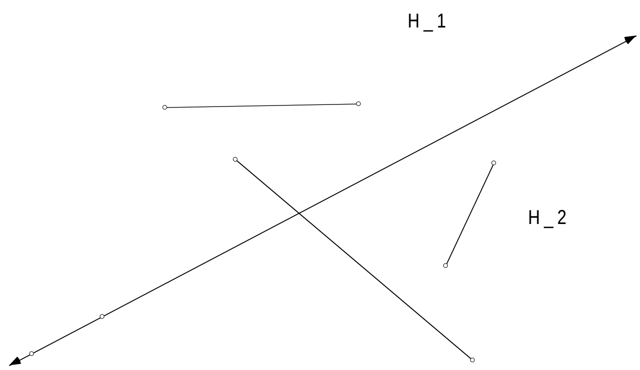

FIGURE 5.1. Plane separation axiom

two points contained in different half-planes *are on opposite sides of* L. One calls L the *boundary* of each of the half-planes. The union of either of the half-planes with L is called a *closed half-plane.*

Exercise 5.4. Let L be a line in a plane $\mathcal{P}$, and let A, B be points of $\mathcal{P}$ which are not on L. Then L intersects the segment $\overline{AB}$ if, and only if, A and B are on the same side of L.

Theorem 5.5. *(Pasch's Axiom) Let $\triangle ABC$ be a triangle in a plane $\mathcal{P}$. Let $L \neq \overleftrightarrow{AB}$ be a line in $\mathcal{P}$ which intersects the segment $\overline{AB}$ at a point between A and B. Then L intersects one of the other two sides of the triangle.*

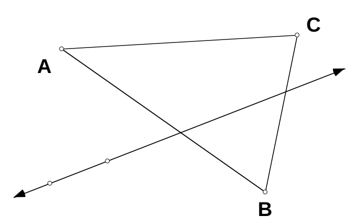

FIGURE 5.2. Pasch's Axiom

Proof. Since L intersects the segment $\overline{AB}$ at a point between A and B, A and B lie on opposite sides of L, by the previous exercise. Suppose that L does not intersect $\overline{AC}$; then A and C are on the same side of L, again, by the previous exercise. It follows that C and B are on opposite sides of L, and therefore L intersects $\overline{CB}$ by the Plane Separation Axiom. $\square$

Remark 5.6. This is called Pasch's *axiom* because it was introduced by Pasch as an axiom, in place of the Plane Separation Axiom. For us, it is a theorem.

Theorem 5.7. *Let $\triangle ABC$ be a triangle in a plane $\mathcal{P}$. Let L be a line in $\mathcal{P}$ which does not contain any of the vertices A, B, C of the triangle. Then L does not intersect all three sides of the triangle.*

Proof. Refer to the figure for Pasch's axiom. Suppose L intersects two of the sides of the triangle, say $\overline{AB}$ and $\overline{BC}$. It has to be show that L does not intersect $\overline{AC}$. Because L intersects $\overline{AB}$, it follows that A and B are on opposite sides of L. Similarly, C and B are on opposite sides of L. Therefore, A and C are on the same side of L, so L does not intersect $\overline{AC}$. □

Theorem 5.8. *Let $\mathcal{P}$ be a plane, and let L be a line in $\mathcal{P}$. Let $M \neq L$ be another line in $\mathcal{P}$ which intersects L. Then M intersects both half-planes of $\mathcal{P}$ determined by L.*

Proof. Let A be the unique point of intersection of L and M (using Theorem 1.1). Let f be a coordinate function on M and let B and C be points on M such that $f(B) < f(A) < f(C)$. Then we have $B — A — C$. Suppose B and C are on the same side of L, and let H denote the half-plane which contains both of them. Since H is convex, the segment $\overline{BC}$ is a subset of H. Since $A \in \overline{BC}$, it follows that $A \in H$. But A is also in L, so $A \in H \cap L = \emptyset$. This contradiction shows that B and C are on opposite sides of L, and thus M intersects both half-planes determined by L. □

Lemma 5.9. *The set of points on a ray, other than the endpoint, is convex.*

Proof. Let $\overrightarrow{AB}$ be a ray, and let S denote $\overrightarrow{AB} \setminus \{A\}$. It must be shown that S is convex. Write M for $\overleftrightarrow{AB}$. Let f be a coordinate function on M such that $f(A) = 0$ and $f(B) > 0$ (Theorem 2.5). Then the ray $\overrightarrow{AB}$ is the set of points C on M such that $f(C) \geq 0$ (Theorem 3.12) and S is the set of points C on M such that $f(C) > 0$. Let C and D be two distinct points in S, and suppose without loss of generality that $0 < f(C) < f(D)$. If $C — X — D$, then $f(C) < f(X) < f(D)$. But then $f(X) > 0$, so $X \in S$. □

Theorem 5.10. *Let $\mathcal{P}$ be a plane, let L be a line in $\mathcal{P}$. Let H be one of the half-planes of $\mathcal{P}$ determined by L. Let A be a point on L and let B be a point in H. Then every point of the $\overrightarrow{AB}$ other than A is an element of H. That is, $\overrightarrow{AB} \setminus \{A\} \subseteq H$. Moreover, $\overleftrightarrow{AB} \cap H = \overrightarrow{AB} \setminus \{A\}$.*

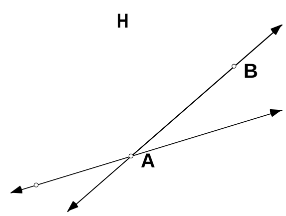

FIGURE 5.3. Theorem 5.10

Proof. Let S denote $\overrightarrow{AB} \setminus \{A\}$. It must be shown that $S = \overleftrightarrow{AB} \cap H$.

Write M for $\overleftrightarrow{AB}$; since $B \notin L$, we know $M \neq L$, and therefore A is the unique point on $M \cap L$. It follows that $S \cap L = \emptyset$.

Let H' denote the half-plane opposite to H. Suppose (in order to reach a contradiction) that $S \cap H'$ contains a point C. According to the previous lemma, S is convex; since both B and C are in S, one has $\overline{BC} \subseteq S$, so $\overline{BC} \cap L \subseteq S \cap L = \emptyset$. On the other hand, by the Plane Separation Axiom, $\overline{BC} \cap L \neq \emptyset$. This contradiction shows that $S \cap H' = \emptyset$. It follows that $S \subseteq H$, so $S \subseteq H \cap \overleftrightarrow{AB}$.

To finish the proof, it must be shown that $H \cap \overleftrightarrow{AB} \subseteq S$, or, equivalently, $\overleftrightarrow{AB} \setminus S \subseteq P \setminus H$. So let $X \in \overleftrightarrow{AB} \setminus S$. If $X = A$, then $X \in L \subseteq P \setminus H$. If $X \neq \overrightarrow{AB}$, then one has $X — A — B$. But then L intersects $\overline{XB}$ at A, so X and B are on opposite sides of L. Hence $X \notin H$. □

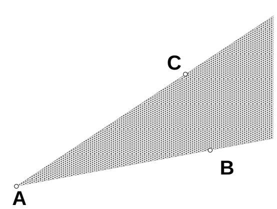

FIGURE 5.4. Angle interior

Definition 5.11. Consider an angle $\angle ABC$ in a plane P. The point B lies in one half-plane H of P determined by $\overleftrightarrow{AC}$. Similarly, the point C lies in one half-plane K of P determined by $\overleftrightarrow{AB}$. The intersection $H \cap K$ of these two half-planes is called the *interior of the angle*. We will call the union of the angle and its interior *the closed wedge determined by the angle*. See figure 5.4.

Theorem 5.12. *Consider an angle $\angle BAC$, and let D be a point in the interior of the angle. Then every point of the ray $\overrightarrow{AD}$, except for the endpoint A, lies in the interior of the angle. That is, $\overrightarrow{AD} \setminus \{A\}$ lies in the interior of the angle. Moreover, the intersection of the line $\overleftrightarrow{AD}$ and the interior of the angle is $\overrightarrow{AD} \setminus \{A\}$.*

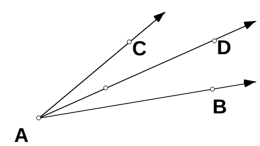

FIGURE 5.5. Theorem 5.12

Proof. This follows from two applications of Theorem 5.10. See Figure 5.5. □

Theorem 5.13. *Consider a triangle $\triangle ABC$. All the points of the segment $\overline{BC}$, except for the endpoints, lie in the interior of the angle $\angle BAC$.*

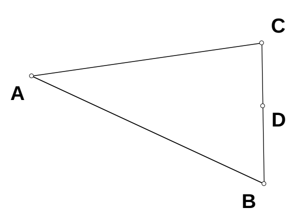

FIGURE 5.6. Theorem 5.13

Proof. See Figure 5.6. Let D be a point between B and C. Then D and B are on the same side of line $\overleftrightarrow{AB}$ because that line intersects $\overleftrightarrow{CD}$ at B, which is not between C and D. Similarly, D and B are on the same side of line $\overleftrightarrow{AC}$. But this means that D is in the interior of angle $\angle CAB$. □

Theorem 5.14. (Crossbar Theorem) *Let $\triangle ABC$ be a triangle, and let D be a point in the interior of the angle $\angle A$. Then the ray $\overrightarrow{AD}$ intersects the side $\overline{BC}$ of the triangle opposite to $\angle A$.*

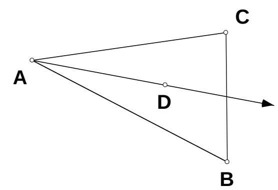

FIGURE 5.7. Crossbar Theorem

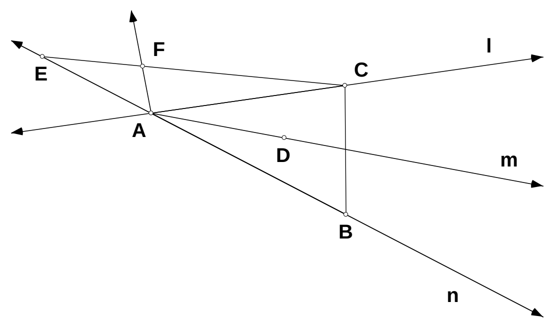

FIGURE 5.8. Crossbar Proof

Proof. Refer to Figure 5.7 for the theorem statement and Figure 5.8 for the proof. This is pretty tricky, and the reader is invited to skip it for now, unless possessed by particular zeal.

Designate the lines $\overleftrightarrow{AC}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{AB}$ by ℓ, m, and n respectively. Let E be a point on line n such that $E — A — B$ (Theorem 3.21). Let F be a point on the segment $\overline{EC}$ such that $E — F — C$ (Theorem 3.21).

We make several observations:

1. E and B are on opposite sides of ℓ because ℓ intersects $\overline{EB}$ at A.
2. E and F are on the same side of ℓ because ℓ intersects $\overleftrightarrow{EF}$ at C, which is not between E and F.
3. D and B are on the same side of ℓ because D is in the interior of the angle $\angle CAB$.
4. Therefore F and D are on opposite sides of ℓ.
5. D and C are on the same side of n since D is in the interior of the angle $\angle CAB$.
6. C and F are on the same side of n because n intersects $\overleftrightarrow{FC}$ at E, which is not between F and C.
7. Therefore F and D are on the same side of n.

Since F and D lie on opposite sides of ℓ, the segment $\overline{FD}$ intersects ℓ at some point A'. Since F and D lie on the same side of n, the point A' is not on n, and in particular $A' \neq A$. If F were on line m, then $\overleftrightarrow{FD}$ would be equal to m. But this cannot be so, beause $\overleftrightarrow{FD}$ intersects ℓ at A' while m intersects ℓ at A.

Thus we conclude that m does not intersect $\overline{EC}$ at any point F between E and C. Thus E and C are on the same side of m. But E and B are on opposite sides of m because m intersects $\overline{EB}$ at A and E — A — B. Therefore C and B are on opposite sides of m, and m must intersect $\overline{CB}$ at some point X between C and B.

It remains only to show that X is on the ray $\overrightarrow{AD} \subseteq m$. But according to Theorem 5.13, X is in the interior of the angle $\angle BAC$, and according to Theorem 5.12, the intersection of the interior of the angle and the line m is contained in the ray $\overrightarrow{AD}$. Therefore X is on the ray $\overrightarrow{AD}$. □

Exercise 5.15. The intersection of two convex sets is convex. The intersection of several convex sets is convex.

In the following, L is a line in a plane $\mathcal{P}$, and H_1 and H_2 are the two half-planes of $\mathcal{P}$ determined by L.

Exercise 5.16. The closed half-plane $H_1 \cup L$ is convex.

Exercise 5.17. H_1 contains at least 3 non-colinear points.

Exercise 5.18. $\mathcal{P}$ is the unique plane containing H_1.