Mathematics 41 Final Exam – F. Goodman
December, 1997
Version 1

General instructions: This exam has 4 pages and 5 problems. There are tables of possibly useful information at the end of the exam. Write answers to all problems in your exam booklet. Your responses will be judged for clarity and completeness as well as correctness, so please provide information about what you are doing, for example: "Now I will find the constants c_1 and c_2 by using the initial conditions."

1. Solve the differential equation

$$(xy + y - 1)\,dx + x\,dy = 0.$$

2. For each of the following differential equations, specify a method by which the equation can be solved, for example: "The equation can be solved as a homogeneous equation." Do not actually solve the equations.

(a)

$$x\,dy = (y + \sqrt{x^2 - y^2})\,dx.$$

(b)

$$\tan x\,dy + \tan y\,dx = 0.$$

(c)

$$(t^2 + y^2)\,\frac{dy}{dt} + 2t(y + 2t) = 0.$$

(d)

$$\frac{y^2 - 3x}{y}\,\frac{dy}{dx} = 1.$$

3. Solve the following initial value problem, following the techniques of Chapter 3. That is, find the general solution to the correponding homogeneous equation, some particular solution to the inhomogeneous equation, and then find the particular solution which satisfies the initial conditions.

$$4y'' - 8y' + 3y = te^{3t}, \quad y(0) = 2,\ y'(0) = 1/2.$$

4. **(a)** Solve the initial value problem

$$y'' + 4y' + 4y = \delta(t), \quad y(0) = 0,\ y'(0) = 0,$$

where $\delta(t)$ is the Dirac delta measure, concentrated at $t = 0$.

(b) Let $g(t)$ be a function possessing a Laplace transform $G(s)$. Solve the initial value problem

$$y'' + 4y' + 4y = g(t), \quad y(0) = 2,\ y'(0) = -1,$$

expressing the solution using a convolution integral.

(c) Solve the initial value problem

$$y'' + 4y' + 4y = e^{3t}, \quad y(0) = 2,\ y'(0) = -1,$$

5. What is meant by an *autonomous* first order ordinary differential equation? Consider the logistics equation

$$\frac{dy}{dt} = r(1 - y/K)y,$$

in which r and K are positive constants. Sketch some representative solution curves of this equation, in the (t, y) plane. Explain why $y = K$ and $y = 0$ are respectively stable and unstable equilibrium points. Show how to derive the general solution to the equation.

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}[F](t)$	$F(s) = \mathcal{L}[f](s)$	Domain of $F(s)$
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n \quad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^p \quad (p > -1)$	$\frac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$

More Laplace Transforms

$f(t) = \mathcal{L}^{-1}[F](t)$	$F(s) = \mathcal{L}[f](s)$	Domain of $F(s)$
$\sin(at)$	$\frac{a}{s^2 + a^2}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2 + a^2}$	$s > 0$
$\sinh(at)$	$\frac{a}{s^2 - a^2}$	$s > \lvert a \rvert$
$\cosh(at)$	$\frac{s}{s^2 - a^2}$	$s > \lvert a \rvert$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2 + b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2 + b^2}$	$s > a$
$t^n e^{at} \quad (n \in \mathbb{N})$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t) f(t-c)$	$e^{-cs} F(s)$	
$e^{ct} f(t)$	$F(s-c)$	
$f(ct) \quad (c > 0)$	$\frac{1}{c} F(\frac{s}{c})$	
$\int_0^t f(t-\tau) g(t) d\tau$	$F(s)G(s)$	
$\delta(t-c)$	e^{-cs}	
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$	

Miscellaneous useful information

- The Bernoulli equation has the form:

$$y' + p(t)y = q(t)y^n,$$

  where n is an integer, $n \neq 0$. The Bernoulli equation can be solved by means of the substitution $v = y^{1-n}$.

- A minor in mathematics at the U. of Iowa requires 15 hours of Department of Mathematics course, at least 12 of which must be in advanced courses. Advanced courses are defined to be 22m:27, 22m:28, and courses numbered 22m:50 or higher (with a few exceptions). The director of the undergraduate program in the Department of Mathematics is Professor Jon Simon.

- Walt Kelley drew the comic strip Pogo. Pogo is the possum at the far right of the image. The turtle next to him (with the horn) is Churchy Lafemme.