Math 26, Second Midterm Exam
April 12, 2006

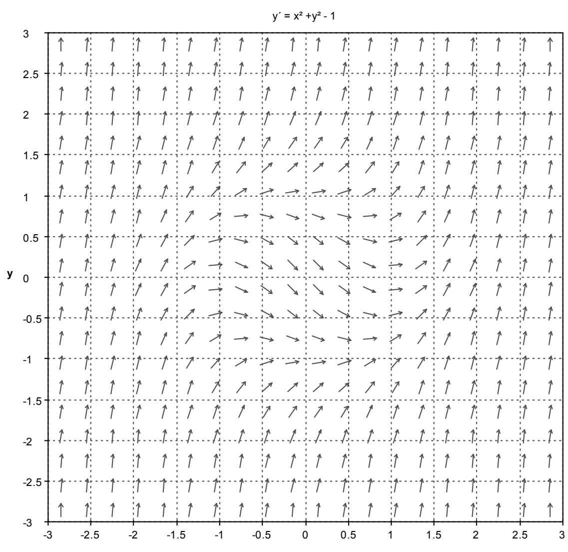
Please show your work, and try to organize the work as coherently as possible. (It is very difficult to assign partial credit if the work is chaotic.) The answer to Exercise 3 is to be recorded on this page, so put your name on this page, and fold this page into your exam book.

(1) Write out the integral expressing the arc length of the curve $y = \sin(x)$, $0 \le x \le \pi$. Do not attempt to evaluate the integral.

(2) A plate in the shape of a right triangle with with legs of length 8 ft. and 3 ft. is submerged in water, with the long edge vertical and the short edge just at the surface of the water. Compute the hydrostatic force on one side of the plate. Denote the weight density of water by δ, and express your answer in terms of δ.

(3) Consider the following direction field corresponding to the DE

$$y' = x^2 + y^2 - 1$$



Sketch the solution to the DE that passes through the point $(0, 0)$. (Draw your sketch right on this test paper.)

(4) A cup of coffee has temperature 95 degrees C. at time 0. The ambient temperature is 20 degrees C. When the temperature reaches 70 degrees C, the coffee is cooling at a rate of 1 degree C per minute. At what time does the coffee reach 70 degrees C.? To answer this, go through the following steps:

(a) Write down the differential equation governing the cooling of the coffee (Newton's law of cooling). In your equation, let $u(t)$ denote the temperature of the coffee at time t. The DE contains an unknown parameter α related to the rate of cooling.
(b) Find the general solution to the DE.
(c) Find the particular solution to the DE with $u(0) = 95$.
(d) Use the additional information to find the parameter α, and, finally, the time t at which $u(t) = 70$.

. . . .

Some formulas.

You do not necessarily need all/any of these formulas, but I promised to include them.

(1) The following formulas come from the double angle formulas:

$$\cos^2(a) = (1/2)(\cos(2a) + 1), \text{ and } \sin^2(a) = (1/2)(1 - \cos(2a)).$$

(2)

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + c,$$
$$\int \csc(x)\,dx = -\ln|\csc(x) + \cot(x)| + c.$$

(3) These come from the angle addition formulas:

$$\sin(a)\cos(b) = (1/2)\left(\sin(a-b) + \sin(a+b)\right)$$
$$\sin(a)\sin(b) = (1/2)\left(\cos(a-b) - \cos(a+b)\right)$$
$$\cos(a)\cos(b) = (1/2)\left(\cos(a-b) + \cos(a+b)\right)$$

(4) The logistic DE is

$$\frac{dP}{dt} = \alpha P(1 - \frac{P}{K}).$$

This is separable, with solution:

$$P(t) = \frac{K}{1 + Ae^{-\alpha t}},$$

where $A = \frac{K-P_0}{P_0}$ and P_0 is the initial population.

(5) A linear differential equation is one of the form:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

The idea for finding a solution is to multiply both sides by an *integrating factor* $I(x)$ chosen so that the left side become a perfect derivative:

$$I(x)(\frac{dy}{dx} + P(x)y) = \frac{d}{dx}(I(x)y).$$

This gives a DE for $I(x)$, namely

$$I'(x) = I(x)P(x).$$

The general solution to this is

$$I(x) = A\exp(\int P(x)dx).$$

Once you know $I(x)$, the solution to the original DE is

$$y = \frac{1}{I(x)} \int I(x)Q(x)dx + C.$$