Math 26, First Midterm Exam
February 22, 2006

Please show your work, and try to organize the work as coherently as possible. (It is very difficult to assign partial credit if the work is chaotic.)

Evaluate the integrals in exercises 1–3. If you use a substitution, say explicitly what substitution you are using. If you do integration by parts, say explicitly what are the parts.

(1) $\int xe^{3x}dx$.

(2) $\displaystyle\int \frac{1}{x\sqrt{4-x^2}}\,dx$.

(3) $\displaystyle\int \sin^2(x)\cos^3(x)\,dx$.

(4) The rational function

$$f(x) = \frac{x}{(x^2+4)(x-1)^2}$$

has the partial fractions expansion

$$-\frac{3x}{25\,(x^2+4)} + \frac{3}{25(x-1)} - \frac{8}{25\,(x^2+4)} + \frac{1}{5(x-1)^2}.$$

Compute $\int f(x)dx$.

(5) Consider the integral

$$\int_1^6 (1/x^2)\sqrt{1+4/x^6}\,dx.$$

The fourth derivative of the integrand

$$f(x) = (1/x^2)\sqrt{1+4/x^6}$$

is decreasing (and positive) on the interval $[1,6]$, and the value of the fourth derivative at $x=1$ is a little less than 3363. How many subintervals (an even number!) do you have to use in Simpson's rule to obtain an approximation to the integral with an error less than 10^{-5}?

Please don't just write down an answer, even though you use your calculator for this one. Write the formulas you are using and indicate what values you are assigning to the parameters in the formulas.

(6) Determine whether the integral

$$\int_1^{\infty} (1/x^2)\sqrt{1+4/x^6}\,dx$$

converges or diverges, by an appropriate comparison. (*Hint:* There is an "obvious" function to compare with, but the inequality goes the wrong way! You have to adjust the comparison function a bit to get an inequality that goes the right way.)

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Some formulas.

(1) The following formulas come from the double angle formulas:

$$\cos^2(a) = (1/2)(\cos(2a)+1), \text{ and } \sin^2(a) = (1/2)(1-\cos(2a)).$$

(2)

$$\int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + c,$$

$$\int \csc(x)\,dx = -\ln|\csc(x)+\cot(x)| + c.$$

(3) These come from the angle addition formulas:

$$\sin(a)\cos(b) = (1/2)\left(\sin(a-b)+\sin(a+b)\right)$$
$$\sin(a)\sin(b) = (1/2)\left(\cos(a-b)-\cos(a+b)\right)$$
$$\cos(a)\cos(b) = (1/2)\left(\cos(a-b)+\cos(a+b)\right)$$

(4) The trapezoid rule approximation for $\int_a^b f(x)dx$ is

$$\frac{\Delta(x)}{2}\left(y_0+2y_1+2y_2+\cdots+2y_{n-1}+y_n\right),$$

where $\Delta(x) = (b-a)/n$, and $y_i = f(x_i)$. The error estimate for the trapezoid rule is

$$|\text{error}| \leq \frac{K(b-a)^3}{12\,n^2},$$

where K is an upper bound for $|f^{(2)}(x)|$ on $[a,b]$.

(5) Simpson's rule approximation for $\int_a^b f(x)dx$ is

$$\frac{\Delta(x)}{3}\left(y_0+4y_1+2y_2+4y_3+\cdots+2y_{n-2}+4y_{n-1}+y_n\right),$$

where $\Delta(x) = (b-a)/n$, $y_i = f(x_i)$, and n is even. The error estimate for Simpson's rule is

$$|\text{error}| \leq \frac{K(b-a)^5}{180\,n^4},$$

where K is an upper bound for $|f^{(4)}(x)|$ on $[a,b]$.