Math 26, First Midterm Exam Review
February, 2006

On the exam, please show your work, and try to organize the work as coherently as possible. (It is very difficult to assign partial credit if the work is chaotic.)

1. Some formulas you know any time day or night.

Here are some formulas that you know, and that will *not* be given to you

(1) The integral formulas on page 475 should be known. Note that the trig formulas come in pairs for functions and "co-functions". For example,

$$\int \sec(x)\tan(x)dx = \sec(x) + C \quad \text{and}$$
$$\int \csc(x)\cot(x)dx = -\csc(x) + C.$$

In each case the second formula is obtained from the first by changing each function to the co-function and changing the sign of the answer. This is also true for the last of the pairs since

$$\int \cot(x)dx = \ln|\sin(x)| + C = -\ln|\csc(x)| + C.$$

(a) Note that you don't need the formula for the integral of a^x, because $a^x = e^{\ln(a)x}$. Just make the substitution $u = \ln(a)x$.

(b) You don't need the last two formulas in the form given, because the substitution $x = au$ reduces the integrals to the simpler forms

$$\int \frac{1}{1+u^2}\,du = \arctan(u) + c, \quad \text{and}$$
$$\int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin(u) + c.$$

(2) The basic trig formula is $\cos^2(x) + \sin^2(x) = 1$. Dividing by $\cos^2(x)$ gives you a formula involving the squares of $\tan(x)$ and $\sec(x)$, and similarly dividing by $\sin^2(x)$ gives you a formula involving the squares of $\cot(x)$ and $\csc(x)$.

(3) The *angle addition formulas* for $\sin(a+b)$ and $\cos(a+b)$.

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b),$$
$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a).$$

(4) From the angle addition formulas, you can immediately get *double angle formulas*

$$\begin{aligned}\cos(2a) &= \cos^2(a) - \sin^2(a)\\ &= 2\cos^2(a) - 1\\ &= 1 - 2\sin^2(a),\\ \sin(2a) &= 2\sin(a)\cos(a).\end{aligned}$$

2. Some formulas I will give you for the exam.

(1) The following formulas come from the double angle formulas, but I will give them to you:

$$\cos^2(a) = (1/2)(\cos(2a) + 1), \text{ and } \sin^2(a) = (1/2)(1 - \cos(2a)).$$

(2)

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + c,$$
$$\int \csc(x)\,dx = -\ln|\csc(x) + \cot(x)| + c.$$

(3) The formulas in the box on the bottom of page 487.

$$\begin{aligned}\sin(a)\cos(b) &= (1/2)\left(\sin(a-b) + \sin(a+b)\right)\\ \sin(a)\sin(b) &= (1/2)\left(\cos(a-b) - \cos(a+b)\right)\\ \cos(a)\cos(b) &= (1/2)\left(\cos(a-b) + \cos(a+b)\right)\end{aligned}$$

(4) The trapezoid rule approximation for $\int_a^b f(x)dx$ is

$$\frac{\Delta(x)}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right),$$

where $\Delta(x) = (b-a)/n$, and $y_i = f(x_i)$. The error estimate for the trapezoid rule is

$$|\text{error}| \leq \frac{K(b-a)^3}{12\,n^2},$$

where K is an upper bound for $|f^{(2)}(x)|$ on $[a,b]$.

Note: I had the error estimates for the midpoint and trapezoid rules mixed up in class. One has a 12 in the denominator, the other a 24.

(5) Simpson's rule approximation for $\int_a^b f(x)dx$ is

$$\frac{\Delta(x)}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right),$$

where $\Delta(x) = (b-a)/n$, $y_i = f(x_i)$, and n is even. The error estimate for Simpson's rule is

$$|\text{error}| \leq \frac{K(b-a)^5}{180\, n^4},$$

where K is an upper bound for $|f^{(4)}(x)|$ on $[a,b]$.

3. Some practice exercises

(1) Evaluate the following integrals. If you use a substitution, say explicitly what substitution you are using. If you do integration by parts, say explicitly what are the parts.

(a) $\displaystyle\int \frac{1}{x(1+\sqrt[3]{x})}\, dx$

(b) $\displaystyle\int \tan^3(x)\, dx$

(c) $\int xe^{-2x} dx$.

(d) $\int x^2 \ln(x) dx$.

(e) $\displaystyle\int \sin^3(x)\cos^4(x)\, dx$.

(f) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\, dx$.

(g) $\displaystyle\int \frac{x^3}{(x^2+4)^{3/2}}\, dx$.

(2) Given the *form* of the partial fractions expansion of the following rational functions, with unknown coefficients $A, B, \ldots$. Do *not* find A, B, etc. For example,

$$\frac{1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}.$$

(a) $\displaystyle\frac{x^2+2x}{(x-1)^2(x^2-4)}$.

(b) $\displaystyle\frac{x^2+2x}{(x-1)\,(x^2+4)^2}$.

(3) The rational function $f(x) = \dfrac{x^2+2x}{(x^2+4)(x-1)^2}$ has the partial fractions expansion

$$\frac{x^2+2x}{(x^2+4)(x-1)^2} = -\frac{14x}{25\,(x^2+4)} + \frac{14}{25(x-1)} - \frac{4}{25\,(x^2+4)} + \frac{3}{5(x-1)^2}.$$

Compute $\int f(x)dx$.

(4) I could ask a question about Simpson's rule or the trapezoid rule of the following sort. You would be given a particular function $f(x)$ and a particular interval $[a, b]$. You would be given the formula for the error bound for the numerical method. Finally you would be given the graph of the appropriate derivative (2nd or 4th derivative) of $f(x)$ over the interval $[a, b]$. From this data you would be asked what n you have to use to make sure that the error in the approximate integral is less than a certain given amount, say 10^{-6}.

(5) Use appropriate comparisons to find out whether each integral is convergent or divergent.

(a) $\displaystyle\int_1^\infty \frac{\sin^2(x)}{1+x^2}\,dx$.

(b) $\displaystyle\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$.

(c) $\displaystyle\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$.

(6) Evaluate $\displaystyle\int_2^\infty xe^{-x}\,dx$.