**Math 26, Final Exam Review**
**May, 2006**

On the exam, please show your work, and try to organize the work as coherently as possible. (It is very difficult to assign partial credit if the work is chaotic.)

## 1. Some formulas you should know.

Here are some formulas that you know, and that will *not* be given to you

(1) The integral formulas on page 475 should be known. Note that the trig formulas come in pairs for functions and "co-functions". For example,

$$\int \sec(x)\tan(x)dx = \sec(x) + C \quad \text{and}$$
$$\int \csc(x)\cot(x)dx = -\csc(x) + C.$$

In each case the second formula is obtained from the first by changing each function to the co-function and changing the sign of the answer. This is also true for the last of the pairs since

$$\int \cot(x)dx = \ln|\sin(x)| + C = -\ln|\csc(x)| + C.$$

(a) Note that you don't need the formula for the integral of $a^x$, because $a^x = e^{\ln(a)x}$. Just make the substitution $u = \ln(a)x$.

(b) You don't need the last two formulas in the form given, because the substitution $x = au$ reduces the integrals to the simpler forms

$$\int \frac{1}{1+u^2}\,du = \arctan(u) + c, \quad \text{and}$$
$$\int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin(u) + c.$$

(2) The basic trig formula is $\cos^2(x) + \sin^2(x) = 1$. Dividing by $\cos^2(x)$ gives you a formula involving the squares of $\tan(x)$ and $\sec(x)$, and similarly dividing by $\sin^2(x)$ gives you a formula involving the squares of $\cot(x)$ and $\csc(x)$.

(3) The *angle addition formulas* for $\sin(a+b)$ and $\cos(a+b)$.

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b),$$
$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a).$$

(4) From the angle addition formulas, you can immediately get *double angle formulas*

$$\begin{aligned}\cos(2a) &= \cos^2(a) - \sin^2(a)\\ &= 2\cos^2(a) - 1\\ &= 1 - 2\sin^2(a),\\ \sin(2a) &= 2\sin(a)\cos(a).\end{aligned}$$

(5) The arc length of a curve $y = f(x)$ from $x = a$ to $x = b$ is

$$\int_a^b \sqrt{1 + f'(x)^2}\,dx = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\,dx$$

(6) Area of a surface of revolution. A surface obtained by rotating a curve $y = f(x)$ from $x = a$ to $x = b$ about the $x$–axis is

$$\int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\,dx = \int_a^b 2\pi y\sqrt{1 + (\frac{dy}{dx})^2}\,dx,$$

assuming $f(x) \geq 0$. For a curve rotated around the $y$–axis, you have to reverse the roles of $x$ and $y$.

(7) The pressure exerted by a fluid at depth $h$ is

$$P = \delta h,$$

where $\delta$ is the weight density of the fluid. The hydrostatic force on a little bit of surface at depth $h$ is

$$\Delta F = P\,\Delta A = \delta h\,\Delta A.$$

The force on a vertically oriented surface extending from depth $a$ to depth $b$ is

$$\delta \int_a^b x\,w(x)\,dx,$$

where $w(x)$ is the width of the surface at depth $x$.

(8) Consider a plate of uniform density in the $x$–$y$ plane. The center of mass of the plate has coordinates $(\bar{x}, \bar{y})$. The $x$–coordinate of the CM is computed as follows: First, the $x$–moment of mass is

$$x\text{–moment} = \int_a^b x\,h(x)\,dx,$$

where the plate lies between $x = a$ and $x = b$, and $h(x)$ is the height of the plate at $x$. In particular, if the plate lies between the $x$–axis and a curve $y = f(x)$ from $x = a$ to $x = b$, then

$$x\text{–moment} = \int_a^b x\,f(x)\,dx,$$

Then,

$$\bar{x} = \frac{x\text{–moment}}{\text{area}}.$$

The formula given above corresponds to slicing the plane vertically. Sometimes you get a better integral to evaluate if you slice horizontally intead. Let's say the plate lies between curves $x = a(y)$ and $x = b(y)$ from $y = c$ to $y = d$. Then

$$x\text{–moment} = \int_c^d \frac{1}{2}(a(y) + b(y))(b(y) - a(y))\,dy.$$

This formula is explained as follows: Slice the plate horizontally. A slice between $y$ and $y + \Delta y$ is approximately a rectangle reaching from $x = a(y)$ to $x = b(y)$ with height $\Delta y$. The area of the slice is $\Delta A = (b(y) - a(y))\Delta y$. The CM of the slice is at $x = (1/2)(b(y) + a(y))$. The slice contributes to the $x$–moment as if its entire mass were at the CM, so its contribution is

$$(1/2)(b(y) + a(y))\Delta A = (1/2)(b(y) + a(y))(b(y) - a(y))\Delta y.$$

For example, if the plate lies between the $y$–axis ($x = 0$) and the curve $x = b(y)$, then

$$x\text{–moment} = \int_c^d \frac{1}{2} b(y)^2\,dy.$$

For the $y$–moment and $\bar{y}$, you have to reverse the roles of $x$ and $y$.

(9) A probability density function $f(x)$ must satisfy $f(x) \geq 0$ and

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

The probability assigned to an interval $[a, b]$ is then

$$\int_a^b f(x)dx.$$

The average or mean of the probability distribution is

$$\int_{-\infty}^{\infty} xf(x)dx.$$

(10) The general solution for a separable differential equation

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

is

$$\int h(y)\,dy = \int g(x)\,dx + c.$$

In particular the differential equation for exponential growth or decay

$$\frac{dy}{dx} = ky$$

is of this sort, and yields the solution $y = Ke^{kt}$.

(11) Newton's law of cooling (or warming) says that the rate at which the temperature of an object changes (when placed in a hot or cold environment) is proportional to the difference between the temperature of the object and the ambient temperature of the environment.

This translates into a formula as follows: Let $u(t)$ be the temperature of the object at time $t$. Let $u_a$ be the ambient temperature. Then

$$\frac{du}{dt} = \alpha(u(t) - u_a),$$

where $\alpha$ is a (possibly unknown) proportionality constant. This is a separable equation. The proportionality constant $\alpha$ is negative.

*In general, you should be able to translate between the verbal description of a differential equation model and the corresponding symbolic equation, as illustrated in the previous two paragraphs.*

(12) Be able to model a mixing problem as in Example 6, page 606.

(13) The slope of a parametric curve $(x(t), y(t))$ is $y'(t)/x'(t)$. The arclength of a parametric curve $(x(t), y(t))$ from $t = a$ to $t = b$ is

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

## 2. Some methods and concepts you should know.

(1) Integration by parts.
(2) Tricks for trig integrals (section 7.2).
(3) Trigonometric substitution (section 7.3).
(4) The method of partial fractions for integration of rational functions. (I would not give an exercise where you have to do a big computation of simultaneous linear equations to compute the partial fractions expansion, but you should understand the concepts.)
(5) Approximate integration and error estimates. (Some formulas will be provided, see below).
(6) Improper integrals, comparison test for improper integrals.
(7) What is a differential equation? What is an initial value problem? What is a solution to a differential equation? What is the geometric interpretation of a differential equation.
(8) Separable differential equations.
(9) Linear differential equations (formulas given below).
(10) Exponential growth and decay.
(11) Logistics equation (formulas given below).
(12) Idea of a parametric curve.

(13) Polar coordinates and curves in polar coordinates.
(14) Sequences and convergence of sequences.
(15) Tests for convergence of series:
  (a) Geometric series.
  (b) Comparison test.
  (c) Integral test.
  (d) Ratio comparision test.
  (e) Ratio test.
  (f) $n$–th root test.
  (g) Alternating series test.
(16) An absolutely convergent series converges.
(17) Error estimates for series.

## 3. Some formulas I will give you for the exam.

(1) The following formulas come from the double angle formulas, but I will give them to you:

$$\cos^2(a) = (1/2)(\cos(2a) + 1), \text{ and } \sin^2(a) = (1/2)(1 - \cos(2a)).$$

(2)

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + c,$$
$$\int \csc(x)\,dx = -\ln|\csc(x) + \cot(x)| + c.$$

(3) The formulas in the box on the bottom of page 487.

$$\sin(a)\cos(b) = (1/2)\left(\sin(a-b) + \sin(a+b)\right)$$
$$\sin(a)\sin(b) = (1/2)\left(\cos(a-b) - \cos(a+b)\right)$$
$$\cos(a)\cos(b) = (1/2)\left(\cos(a-b) + \cos(a+b)\right)$$

(4) The trapezoid rule approximation for $\int_a^b f(x)dx$ is

$$\frac{\Delta(x)}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right),$$

where $\Delta(x) = (b-a)/n$, and $y_i = f(x_i)$. The error estimate for the trapezoid rule is

$$|\text{error}| \leq \frac{K(b-a)^3}{12\,n^2},$$

where $K$ is an upper bound for $|f^{(2)}(x)|$ on $[a, b]$.

(5) Simpson's rule approximation for $\int_a^b f(x)dx$ is

$$\frac{\Delta(x)}{3}\left(y_0+4y_1+2y_2+4y_3+\cdots+2y_{n-2}+4y_{n-1}+y_n\right),$$

where $\Delta(x)=(b-a)/n$, $y_i=f(x_i)$, and $n$ is even. The error estimate for Simpson's rule is

$$|\text{error}|\leq\frac{K(b-a)^5}{180\,n^4},$$

where $K$ is an upper bound for $|f^{(4)}(x)|$ on $[a,b]$.

(6) The logistic DE is

$$\frac{dP}{dt}=kP(1-\frac{P}{K}).$$

This is separable, with solution:

$$P(t)=\frac{K}{1+Ae^{-kt}},$$

where $A=\frac{K-P_0}{P_0}$ and $P_0$ is the initial population.

(7) A linear differential equation is one of the form:

$$\frac{dy}{dx}+P(x)y=Q(x).$$

The idea for finding a solution is to multiply both sides by an *integrating factor* $I(x)$ chose so that the left side become a perfect derivative:

$$I(x)(\frac{dy}{dx}+P(x)y)=\frac{d}{dx}(I(x)y).$$

This give a DE for $I(x)$, namely

$$I'(x)=I(x)P(x).$$

The general solution to this is

$$I(x)=A\exp(\int P(x)dx).$$

Once you know $I(x)$, the solution to the original DE is

$$I(x)y=\int I(x)Q(x)dx+C.$$

## 4. Some practice exercises

(1) Evaluate the following integrals. If you use a substitution, say explicitly what substitution you are using. If you do integration by parts, say explicitly what are the parts.

(a) $\displaystyle\int \frac{1}{x(1+\sqrt[3]{x})}\,dx$

(b) $\displaystyle\int \tan^3(x)\,dx$

(c) $\int xe^{-2x}dx$.

(d) $\int x^2\ln(x)dx$.

(e) $\displaystyle\int \sin^3(x)\cos^4(x)\,dx$.

(f) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx$.

(g) $\displaystyle\int \frac{x^3}{(x^2+4)^{3/2}}\,dx$.

(2) Give the *form* of the partial fractions expansion of the following rational functions, with unknown coefficients $A, B, \ldots$. Do *not* find $A, B$, etc. For example,

$$\frac{1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}.$$

(a) $\displaystyle\frac{x^2+2x}{(x-1)^2(x^2-4)}$.

(b) $\displaystyle\frac{x^2+2x}{(x-1)\left(x^2+4\right)^2}$.

(3) The rational function $f(x) = \dfrac{x^2+2x}{(x^2+4)(x-1)^2}$ has the partial fractions expansion

$$\frac{x^2+2x}{(x^2+4)(x-1)^2} = -\frac{14x}{25\left(x^2+4\right)} + \frac{14}{25(x-1)} - \frac{4}{25\left(x^2+4\right)} + \frac{3}{5(x-1)^2}.$$

Compute $\int f(x)dx$.

(4) I could ask a question about Simpson's rule or the trapezoid rule of the following sort. You would be given a particular function $f(x)$ and a particular interval $[a, b]$. You would be given the formula for the error bound for the numerical method. Finally you would be given the graph of the appropriate derivative (2nd or 4th derivative) of $f(x)$ over the interval $[a, b]$. From this data you would be asked what $n$ you have to use to make sure that the error in the approximate integral is less than a certain given amount, say $10^{-6}$.

(5) Use appropriate comparisons to find out whether each integral is convergent or divergent.
(a) $\displaystyle\int_1^\infty \frac{\sin^2(x)}{1+x^2}\,dx.$

(b) $\displaystyle\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx.$

(c) $\displaystyle\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx.$

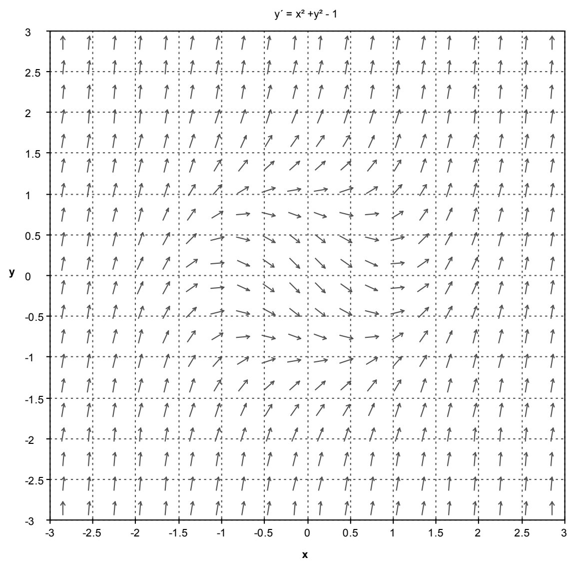
(6) Evaluate $\displaystyle\int_2^\infty xe^{-x}\,dx.$

(7) Write out the integral expressing the arc length of the curve $y = \sin(x)$, $0 \le x \le \pi$. Do not attempt to evaluate the integral.

(8) A plate in the shape of a right triangle with with legs of length 8 ft. and 3 ft. is submerged in water, with the long edge vertical and the short edge just at the surface of the water. Compute the hydrostatic force on one side of the plate. Denote the weight density of water by $\delta$, and express your answer in terms of $\delta$.

(9) Consider the following direction field corresponding to the DE

$$y' = x^2 + y^2 - 1$$




Sketch the solution to the DE that passes through the point $(0,0)$. (Draw your sketch right on this test paper.)

(10) A cup of coffee has temperature 95 degrees C. at time 0. The ambient temperature is 20 degrees C. When the temperature reaches 70 degrees C, the coffee is cooling at a rate of 1 degree C per minute. At what time does the coffee reach 70 degrees C.? To answer this, go through the following steps:

(a) Write down the differential equation governing the cooling of the coffee (Newton's law of cooling). In your equation, let $u(t)$ denote the temperature of the coffee at time $t$. The DE contains an unknown parameter $\alpha$ related to the rate of cooling.

(b) Find the general solution to the DE.

(c) Find the particular solution to the DE with $u(0) = 95$.

(d) Use the additional information to find the parameter $\alpha$, and, finally, the time $t$ at which $u(t) = 70$.

(11) Determine whether each series converges. Make your use of any convergence test explicit.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n-1}{n^2+n}$

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n-1}{n^3-n}$

(d) $\sum_{n=1}^{\infty} \frac{n^2}{5^n}$

(e) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

(f) $\sum_{n=1}^{\infty} n^2 e^{-n}$

(g) $\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!}$