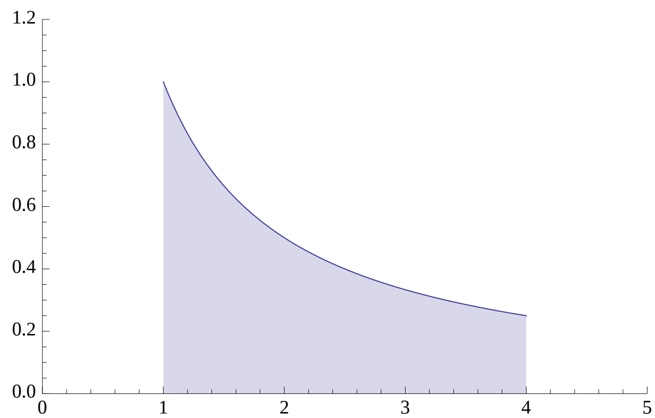
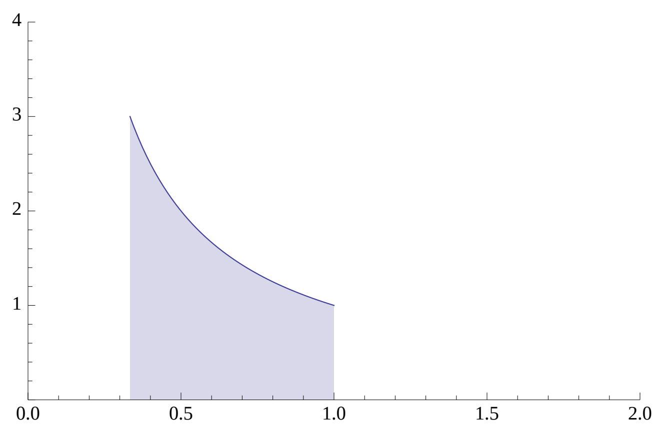
Exponential and logarithm functions

What are the exponential and logarithm functions and what is the number e? There are a number of approaches to developing the exponential and log functions. The easiest, most elegant and satisfactory approach requires, unfortunately, that one already knows the elements of *integral calculus*, the second half of this course. But I will outline the approach for you here.

For $t \geq 1$, define $A(t)$ to be the area under the curve $y = 1/x$ between $x = 1$ and $x = t$. The following picture illustrates $A(4)$.



For $t \leq 1$, define $A(t)$ to be the negative of the area under the curve $y = 1/x$ between $x = t$ and $x = 1$. For example, $A(1/3)$ is the negative of the area illustrated in the following picture:



Then one can show, and this is not all that difficult, that the function $A(t)$ has the properties of a logarithm function, namely

$$A(ab) = A(a) + A(b) \tag{1}$$

for any positive numbers a and b. Moreover, one can show that $\lim_{t\to\infty} A(t) = \infty$ and $\lim_{t\to 0^+} A(t) = -\infty$.

By basic properties of integrals (to be developed later in this course), A is a differentiable function on its entire domain $(0, \infty)$, and $A'(t) = 1/t$. Moreover, again by basic properties of integrals, A is a strictly increasing function whose range is all of the real numbers.

Because A has the properties of a logarithm function, we call it the natural logarithm and denote it by $A(t) = \ln(t)$. *So, in this approach,* ln *is defined in terms of the area under the curve* $y = 1/x$. We have

$$\frac{d}{dt}\ln(t) = 1/t. \tag{2}$$

Because ln is a strictly increasing function from $(0, \infty)$ with range $\mathbb{R}$, it has an inverse function $\exp : \mathbb{R} \to (0, \infty)$. Because ln has the properties of a logarithm function, it is not at all hard to show that exp has the properties of an exponential function, for example

$$\exp(a + b) = \exp(a)\exp(b), \tag{3}$$

for all real numbers a and b.

Because ln and exp are inverse functions, we have $\ln(a) = b$ if, and only if, $\exp(b) = a$. There is a number e such that $\ln(e) = 1$; in fact, e is the unique number such that the area under the curve $y = 1/x$ between $x = 1$ and $x = e$ is equal to 1. Since $\ln(e) = 1$, we have

$$\exp(1) = e.$$

Note also that $\ln(1) = 0$, so

$$\exp(0) = 1.$$

For all real numbers a, we have

$$\exp(a)\exp(-a) = \exp(a + (-a)) = \exp(0) = 1,$$

so

$$\exp(-a) = \frac{1}{\exp(a)}. \tag{4}$$

Now using the basic property of the exponential function (equation (3)), we can show that

$$\exp(m/n) = e^{m/n}$$

for rational numbers m/n, where, on the right hand side, $e^{m/n}$ is defined in the elementary way, as the n–th root of the positive number e^m.

At this point we define e^x as

$$e^x = \exp(x),$$

for all real numbers x. When x is rational, this agrees with the elementary meaning of e^x. Finally, it follows from properties of inverse functions, which we will discuss in class, that e^x is a differentiable function and

$$\frac{d}{dx}e^x = e^x.$$

All of this may seem rather roundabout, but it is simpler than the supposedly elementary approach in the text, which hides a lot of technical difficulties. Here is the outline of the supposedly elementary approach.

(1) Given a positive number $a \neq 1$, one defines $a^{m/n}$ in the elementary way for rational numbers m/n. One observes that this function, so far defined only for rational numbers, has the usual properties of an exponential.
(2) One shows that a^x is a continuous function of the rational variable x. Now one has to show that this function can be extended to a continuous function defined on all real numbers, and that the resulting extension still has the usual properties of an exponential. This involves a lot of technical detail.
(3) For convenience, write $f_a(x) = a^x$. One shows that f_a is differentiable and that $f_a'(x) = f_a'(1) f_a(x)$. That is, the derivative of a^x is a constant multiple of a^x and the constant factor is the derivative of a^x at $x = 1$.
(4) Now one can show that there is a unique positive number e such that the constant factor in the derivative is 1, so $\frac{d}{dx} e^x = e^x$. This is taken as the definition of e.
(5) Finally, ln is defined as the inverse of the function e^x. It follows from properties of inverse functions (to be discussed in class) that $\frac{d}{dx} \ln(x) = 1/x$.

There are a lot of difficulties to be overcome in step (2), so this is actually not such an elementary approach.