Assignment 6

(1) Let F be a field and let V be a countably infinite dimensional vector space over F. Let $A = \mathrm{End}_F(V)$. Let I be the ideal of finite rank transformations,

$$I = \{T \in A : \dim_F(T(V)) < \infty\}.$$

Let $B = A/I$ and let $\pi : A \to B$ be the quotient map.

(a) Show that B is a simple F–algebra.

(b) Construct an infinite decreasing sequence of subspace of V,

$$V = V_0 \supset V_1 \supset V_2 \supset \dots$$

such that V_i/V_{i+1} is infinite dimensional for each i.

(c) Let

$$M_i = \{T \in A : T(V) \subseteq V_i\},$$

and

$$N_i = \{T \in A : \dim_F(T(V_i)) < \infty\}.$$

Show that $(\pi(M_i))_{i\geq 0}$ is an infinite, strictly decreasing sequence of right ideals in B and that $(\pi(N_i))_{i\geq 0}$ is an infinite, strictly increasing sequence of left ideals in B. Thus B is not right Artinian, and not left Noetherian. Conclude that B is not semisimple.

(d) Conclude that B is also not left Artinian and not right Noetherian.

(2) Let M be a left module over a ring R. Define the *socle* of M to be the span of all simple submodules of M. Denote the socle by $\mathrm{soc}(M)$.

(a) Show $\mathrm{soc}(M)$ is a semisimple submodule of M, that every semisimple submodule is contained in $\mathrm{soc}(M)$ and that M is semisimple if, and only if, $M = \mathrm{soc}(M)$.

(b) Show if M is non–zero and Artinian, then $\mathrm{soc}(M) \neq (0)$.

(c) Show that if M is Artinian and $\varphi : M \to N$ is a module homomorphism that is injective on the socle of M, then φ is injective.

(3) Define a partial order on idempotents in a ring R by $e \leq f$ if $ef = fe = e$. An idempotent is called *primitive* or *minimal* if it is minimal in this partial order.

(a) Show $e \leq f$ if, and only if, there is an idempotent e' such that $f = e + e'$ and $ee' = e'e = 0$.

(b) Show that eRe is a ring with identity e. Show that $\mathrm{End}_R(Re)$ is anti-isomorphic to eRe.

(c) Recall that in a semisimple ring R, every left ideal has the form Re for some idempotent e. Show that the following are equivalent for an idempotent in a semisimple ring.

(i) e is primitive.

(ii) Re is a minimal left ideal

(iii) eRe is a division ring.

(4) A non–zero R–module M is said to be *indecomposable* if it is not a direct sum of proper submodules. Let R be the ring of 2–by–2 upper triangular matrices over a field F, and let M be the vector space F^2 regarded as a left R–module. Show that M is indecomposable but not simple as an R–module. Show that $\mathrm{End}_R(M)$ consists of scalar multiplications $x \mapsto \alpha x$ for $\alpha \in F$.

(5) Let R be a ring and M a finitely generated semisimple left R–module. Prove that $\mathrm{End}_R(M)$ is a semisimple ring.

(6) The radical of an R module M to be the intersection of all maximal proper submodules, and the radical of R is the radical of the R–module ${}_RR$. The radical is a submodule and in particular the radical of R is a left ideal.
  (a) Show that if $\varphi : M \to N$ is an R–module homomorphism, then $\varphi(\mathrm{rad}(M)) \subseteq \mathrm{rad}(N)$. In particular $\mathrm{End}_R(R)$ preserves the radical of ${}_RR$.
  (b) Conclude that $\mathrm{rad}({}_RR)$ is a two–sided ideal of R.
  (c) Conclude also that $\mathrm{rad}({}_RR)M \subseteq \mathrm{rad}(M)$.