Assignment 5

R will always denote a unital ring. Modules are left modules, unless otherwise indicated.

(1) Let X, A, B be R–modules and $\alpha : X \to A, \beta : X \to B$. Define $C' = (A \oplus B)/N$, where $N = \{(\alpha(x), -\beta(x)) : x \in X\}$. Define $\alpha' : B \to C'$ by $\alpha'(b) = (0, b) + N$, and $\beta' : A \to C$ by $\beta'(a) = (a, 0) + N$. Show that $\beta' \circ \alpha = \alpha' \circ \beta$. Discover and prove a universal property for the triple (α', β', C') relative to the fixed data (X, A, B, α, β). (Hint: Consider $C'', \alpha" : B \to C'', \beta'' : A \to C"$ such that $\beta" \circ \alpha = \alpha" \circ \beta$.) Show that if α is injective, then α' is injective.

(2) Use the construction in question (1) to show that if an R–module Q has the property that every short exact sequence $0 \to Q \to B \to C \to 0$ splits, then Q is injective.

(3) Invent and analyze an analogue of problem (1) with all the arrows reversed.

(4) Use your solution to exercise (3) to give a new proof that a module P is projective if it has the property that every short exact sequence $0 \to A \to B \to P \to 0$ splits.

(5) Recall that an R–module N is said to be flat if the functor $\_\_ \otimes_R N$ is exact. Let $M = \bigoplus_j M_j$. Show that M is flat if, and only if, each M_j is flat. (You will use that for every A, we have a natural isomorphism $A \otimes M \cong \bigoplus_j (A \otimes M_j)$.) Observe that R is flat as an R module. Conclude that every free R module is flat and that every projective R module is flat.