Assignment 4

(1) Let A and B be rings. We showed in class that for (bi)modules N_A, M_B, and ${}_AQ_B$, there is an isomorphism of abelian groups:

$$\mathrm{Hom}_B(N \otimes_A Q, M) \cong \mathrm{Hom}_A(N, \mathrm{Hom}_B(Q, M)).$$

Switching left and right, we have, for (bi)modules ${}_AN$, ${}_BM$, and ${}_BQ_A$,

$$(*) \quad \mathrm{Hom}_B(Q \otimes_A N, M) \cong \mathrm{Hom}_A(N, \mathrm{Hom}_B(Q, M)).$$

Now suppose that $A \subseteq B$ and that A and B have the same identity. If N is a left A–module, we can get B–modules in two ways from N. The *induced module* is

$$\mathrm{ind}_A^B(N) = B \otimes_A N,$$

where B is regarded as an A–B bimodule. The *co-induced* module is

$$\mathrm{coind}_A^B(N) = \mathrm{Hom}_A(B, N).$$

Given a B–module M, we can regard it as an A module by restriction. The restricted module is denoted $\mathrm{res}_A^B(M)$.

(a) Show that

$$\mathrm{res}_A^B(M) \;\cong\; \mathrm{Hom}_B({}_BB_A, M),$$

as A–modules, and also

$$\mathrm{res}_A^B(M) \;\cong\; {}_AB_B \otimes M,$$

as A–modules.

(b) Apply (*) with appropriate choice of Q to show that

$$\mathrm{Hom}_B(\mathrm{ind}_A^B(N), M) \cong \mathrm{Hom}_A(N, \mathrm{res}_A^B(M)).$$

(c) Apply (*) with the roles of M and N reversed to show that

$$\mathrm{Hom}_B(M, \mathrm{coind}_A^B(N)) \cong \mathrm{Hom}_A(\mathrm{res}_A^B(M), N).$$

(d) Referring to part (b), trace through the various identifications to show that the isomorphism is (more or less) explicitly given as follows. If

$$\varphi \in \mathrm{Hom}_B(B \otimes_A N, M),$$

then the corresponding element of $\mathrm{Hom}_A(N, \mathrm{res}_A^B(M))$ is given as

$$\hat{\varphi}(w) = \varphi(1_B \otimes w).$$

Verify directly that $\hat{\varphi}$ defined in this way is an A–module homomorphism.

(2) Let F be a field and let V be a countably infinite dimensional vector space over F. Let $A = \mathrm{End}_F(V)$. Let I be the ideal of finite rank transformations,

$$I = \{T \in A : \dim_F(T(V)) < \infty\}.$$

Let $B = A/I$ and let $\pi : A \to B$ be the quotient map.

(a) Show that B is a simple F–algebra.

(b) Construct an in finite decreasing sequence of subspace of V,

$$V = V_0 \supset V_1 \supset V_2 \supset \dots$$

such that V_i/V_{i+1} is infinite dimensional for each i.

(c) Let

$$M_i = \{T \in A : T(V) \subseteq V_i\},$$

and

$$N_i = \{T \in A : \dim_F(T(V_i)) < \infty\}.$$

Show that $(\pi(M_i))_{i\geq 0}$ is an infinite, strictly decreasing sequence of right ideals in B and that $(\pi(N_i))_{i\geq 0}$ is an infinite, strictly increasing sequence of left ideals in B. Thus B is not right Artinian, and not left Noetherian.

(d) Conclude that B is also not left Artinian and not right Noetherian.

(e) Conclude that A is not left or right Artinian and not left or right Noetherian.

(3) Let M be a left module over a ring R. Define the *socle* of M to be the span of all simple submodules of M. Denote the socle by $\mathrm{soc}(M)$.

(a) Show $\mathrm{soc}(M)$ is a semisimple submodule of M, that every semisimple submodule is contained in $\mathrm{soc}(M)$ and that M is semisimple if, and only if, $M = \mathrm{soc}(M)$.

(b) Show if M is non–zero and Artinian, then $\mathrm{soc}(M) \neq (0)$.

(c) Show that if M is Artinian and $\varphi : M \to N$ is a module homomorphism that is injective on the socle of M, then φ is injective.

(4) Define a partial order on idempotents in a ring R by $e \leq f$ if $ef = fe = e$. An idempotent is called *primitive* or *minimal* if it is minimal in this partial order.

(a) Show $e \leq f$ if, and only if, there is an idempotent e' such that $f = e + e'$.

(b) Show that eRe is a ring with identity e. Show that $\mathrm{End}_R(Re)$ is anti-isomorphic to eRe.

(c) Recall that in a semisimple ring R, every left ideal has the form Re for some idempotent e. Show that the following are equivalent for an idempotent in a semisimple ring.

(i) e is primitive.

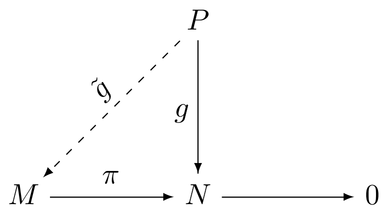
(ii) Re is a minimal left ideal

(iii) eRe is a division ring.

(5) A non–zero R–module M is said to be *indecomposable* if it is not a direct sum of proper submodules. Let R be the ring of upper triangular matrices over a field F, and let M be the vector space F^2 regarded as a left R–module. Show that M is indecomposable but not simple as an R–module. Show that $\mathrm{End}_R(M)$ consists of scalar multiplications $x \mapsto \alpha x$ for $\alpha \in F$.

(6) Let R be a ring and M a finitely generated semisimple left R–module. Prove that $\mathrm{End}_R(M)$ is a semisimple ring.

(7) Recall that we defined an R–module P to be *projective* if whenever $M \xrightarrow{\pi} N \to 0$ is an exact sequence of R modules and $P \xrightarrow{g} N$ is an R–module homomorphism, then g lifts: namely there is a homomorphism $P \xrightarrow{\tilde{g}} M$ such that $g = \pi \circ \tilde{g}$.



In fact P is projective if, and only if, whenever $M \xrightarrow{\pi} P \to 0$ is exact, then there exists $M \xleftarrow{s} P$ such that $\pi \circ s = \mathrm{id}_P$. Show that a ring is semisimple if, and only if, every R–module is projective.

(8) We have defined the radical of an R module M to be the intersection of all maximal proper submodules, and the radical of R to be the radical of the R–module ${}_RR$. The radical is a submodule and in particular the radical of R is a left ideal.
   (a) Show that if $\varphi : M \to N$ is an R–module homomorphism, then $\varphi(\mathrm{rad}(M)) \subseteq \mathrm{rad}(N)$. In particular $\mathrm{End}_R(R)$ preserves the radical of ${}_RR$.
   (b) Conclude that $\mathrm{rad}({}_RR)$ is a two–sided ideal of R.
   (c) Conclude also that $\mathrm{rad}({}_RR)M \subseteq \mathrm{rad}(M)$.