Math 16, Homework 3

1. Write out the truth tables for "A implies B" and for "B implies A" and observe that they are different. (The statement "B implies A" is called the *converse* of "A implies B".)
2. Write out the truth tables for "A implies B" and for "not(B) implies not(A)" and observe that they are the same! (The statement "not(B) implies not(A)" is called the *contrapositive* of "A implies B".)
3. Form the negation of each of the following sentences. Simplify until the result contains negations only of simple sentences.
   (a) Tonight I will go to a restaurant for dinner or to a movie.
   (b) Tonight I will go to a restaurant for dinner and to a movie.
   (c) If today is Tuesday, I have missed a deadline.
   (d) For all lines L, L has at least two points.
   (e) For every line L and every plane $\mathbb{P}$, if L is not a subset of $\mathbb{P}$, then $L \cap \mathbb{P}$ has at most one point.
4. Same instructions as for the previous problem Watch out for implicit universal quantifiers.
   (a) If x is a real number, then $\sqrt{x^2} = |x|$.
   (b) If x is a natural number and x is not a perfect square, then $\sqrt{x}$ is irrational.
   (c) If n is a natural number, then there exists a natural number N such $N > n$.
   (d) If L and M are distinct lines, then either L and M do not intersect, or their intersection contains exactly one point.
5. (a) Let $f(x) = \dfrac{x}{1+x}$. Let f^{*n} denote the n-fold composition of f; thus $f^{*1}(x) = f(x)$, $f^{*2}(x) = f(f(x))$, $f^{*3}(x) = f(f(f(x)))$, and so forth. Show by induction that for all natural numbers n, $f^{*n}(x) = \dfrac{x}{1+nx}$.
   (b) Let a be any positive number and define a sequence with initial value a and updating function f:

$$\begin{cases} a_1 = a \\ a_{n+1} = f(a_n). \end{cases}$$

   Describe the behavior of the sequence a_n.
6. Let a be a positive number. Show by induction that for all natural numbers n, $(1+a)^n \geq 1 + na$.

7. Suppose that the amount y_n of drug present in the body after n daily doses satisfies the updating rule:

$$\begin{cases} x_1 = 500 \\ x_n = 250 + .7\,(1 + .01\cos(\frac{2\pi n}{28}))\; x_{n-1} \quad \text{for } n \geq 2 \end{cases}$$

By comparing this sequence with that defined by

$$\begin{cases} y_1 = 500 \\ y_n = 250 + .707\; y_{n-1} \quad \text{for } n \geq 2, \end{cases}$$

find an *upper bound* for the sequence x_n, i.e., a number M such that $x_n \leq M$ for all natural numbers n. Hint: Show by induction that $x_n \leq y_n$ for all natural numbers n, and find an upper bound for the sequence y_n.

8. Let x_n be as in the previous exercise. Is it possible to find a *lower bound* for the numbers x_n, for $n \geq 50$, i.e. a number m such that $x_n \geq m$ for all natural numbers $n \geq 50$?