Mathematics 90 Final Exam – F. Goodman
December 1999

1. Show that given 52 integers, there exist two of them whose sum or difference is divisible by 100.

2. **(a)** How many ways are there to distribute 3 identical oranges, 5 identical candy canes, and one lump of coal to 3 (distinguishable) holiday stockings?

(b) What if the candy canes are of five distinct flavors?

3. **(a)** Count the permutations of the multiset $\{2a, 2b, 2c\}$.

(b) Count the circular permutations of the multiset $\{2a, 2b, 2c\}$, using Polya-Burnside theory. That is, count the number of orbits of the cyclic group $\mathbb{Z}_6$ acting on multiset permutations.

4. How many integer solutions are there to:

(a) $x_1 + x_2 + x_3 + x_4 = 15, \ \ x_i \geq 0$?

(b) $x_1 + x_2 + x_3 + x_4 \leq 15, \ \ x_i \geq 0$?

(c) $x_1 + x_2 + x_3 + x_4 = 15, \ \ x_i \geq 0, \ x_2 \geq 2, \ x_4 \leq 3$?

5. Give generating function models for each of the parts of problem 4; that is, replace 15 by a variable n, and write the generating function for the number of solutions to the equation or inequality.

6. **(a)** Give the generating function for distributions of r identitical rice krispies into 17 *distiguishable* rice krispy boxes. (This is the same as the generating function for solutions to $x_1 + x_2 + \cdots + x_{17} = r$.)

(b) Give the generating function for distributions of r identitical rice krispies into 17 *indistiguishable* rice krispy boxes. (This is the same as the generating function for solutions to $x_1 + x_2 + \cdots + x_{17} = r$, where the order of the x_i's doesn't matter. Since the order doesn't matter, arrange the x_i's in decreasing order. So you are actually counting *partitions* of r with no more than 17 parts!)

(c) Give the generating function for distributions of r identitical rice krispies into r *indistiguishable* rice krispy boxes. (Now there is no longer any restriction on the number of parts in the partitions.)

Remark: The generating function for partitions of n with parts in a particular subset S of the natural numbers is

$$\prod_{j \in S} \frac{1}{1 - x^j};$$

in particular the generating function for all partitions is

$$\prod_{j=1}^{\infty} \frac{1}{1 - x^j};$$