Mathematics 121 Midterm Exam II – Fred Goodman
April, 2005
Version 2

Do all problems.
Responses will be judged for accuracy, clarity and coherence.

1. Prove that a vector space with a finite spanning set has a finite basis.

2. Let R be a commutative ring with 1. Consider the standard basis $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$ of R^3.

(a) Let $\mu : (R^3)^3 \longrightarrow R^3$ be a multilinear, alternating function. Show that

$$\mu(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) = \det([\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3])\mu(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3),$$

for any $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3 \in R^3$. Here $[\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3]$ is the 3–by–3 matrix whose columns are the "vectors" $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ and the determinant is defined via a sum over the symmetric group S_3.

(b) Show that $\det(AB) = \det(A)\det(B)$ for any two 3–by–3 matrices A, B with entries in R.

3. State, but do not prove, the theorem on the invariant factor decomposition of a finitely generated module over a principal ideal domain.

4. Give the "short" proof of the Cayley-Hamilton theorem, by discussing the relation between the Smith normal form of $x - A$, where A is a square matrix over a field K, the invariant factors of A, the minimal polynomial of A, and the characteristic polynomial of A.

5. Consider the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 3 \\ 1 & 2 & 0 & -4 & 0 \\ 3 & 1 & 2 & -4 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 4 \end{bmatrix}$$

The Smith Normal Form of $x - A$ is

$$D(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1+x & 0 \\ 0 & 0 & 0 & 0 & (-2+x)^3(-1+x) \end{bmatrix}$$

The last diagonal entry of $D(x)$ expands to $x^4 - 7x^3 + 18x^2 - 20x + 8$

(a) Determine the minimal and characteristic polynomials of A.

(b) Write down the Jordan Canonical Form and the Rational Canonical Form of A.

(c) Find a matrix S such that $S^{-1}AS$ is in Jordan form.
The following information is useful for this: One has

$$x - A = P(x)D(x)Q(x),$$

where $P(x)$ and $Q(x)$ are invertible 5–by–5 matrices with entries in $\mathbb{Q}[x]$. The matrix $P(x)^{-1}$ is

$$P(x)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{4}(-1+x) & \frac{1}{4}(-2+x)(-1+x) & -1+3x-x^2 & -\frac{3}{4}(-1+x) \\ \frac{4-x}{3} & 0 & 0 & \frac{4}{3}(-2+x) & 1 \end{bmatrix}$$

The following matrices might also be useful to you:

$$A^2 = \begin{bmatrix} -5 & 0 & 0 & 0 & 9 \\ 1 & 4 & 0 & -12 & 3 \\ 10 & 4 & 4 & -16 & -9 \\ 0 & 0 & 0 & 1 & 0 \\ -6 & 0 & 0 & 0 & 10 \end{bmatrix}, \qquad A^3 = \begin{bmatrix} -13 & 0 & 0 & 0 & 21 \\ -3 & 8 & 0 & -28 & 15 \\ 24 & 12 & 8 & -48 & -18 \\ 0 & 0 & 0 & 1 & 0 \\ -14 & 0 & 0 & 0 & 22 \end{bmatrix}$$

$$A - 2 = \begin{bmatrix} -3 & 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & -4 & 0 \\ 3 & 1 & 0 & -4 & -3 \\ 0 & 0 & 0 & -1 & 0 \\ -2 & 0 & 0 & 0 & 2 \end{bmatrix} \qquad (A-2)^2 = \begin{bmatrix} 3 & 0 & 0 & 0 & -3 \\ -3 & 0 & 0 & 4 & 3 \\ -2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$$(A-2)^3 = \begin{bmatrix} -3 & 0 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -2 & 0 & 0 & 0 & 2 \end{bmatrix} \qquad (A-1) = \begin{bmatrix} -2 & 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & -4 & 0 \\ 3 & 1 & 1 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 3 \end{bmatrix}$$