Mathematics 121 Midterm Exam – Fred Goodman
April, 2006
Version 1

Do all problems.
Responses will be judged for accuracy, clarity and coherence.

1. Show that any two bases of a finite dimensional vector space have the same cardinality.

2.

(a) Show that a linear transformation on a finite dimensional vector space V over a field K determines a finitely generated torsion $K[x]$–module structure on V.

(b) Show that two linear transformations are similar if, and only if, they determine isomorphic $K[x]$– modules.

3. State, but do not prove, the theorem on the invariant factor decomposition of a finitely generated module over a principal ideal domain. (Note that the module is not assumed to ba a torsion module.)

4. Consider the matrix

$$A = \begin{bmatrix} 2 & -4 & -12 & 17 & 12 \\ 0 & -15 & 9 & 68 & 55 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -4 & 0 & 18 & 13 \\ 0 & 0 & 3 & 0 & 2 \end{bmatrix}$$

The characteristic polynomial of A is $\chi_A(x) = (x-1)^2(x-2)^3$.

(a) Find the Jordan canonical form of A and find a matrix S in $\mathrm{Mat}_5(\mathbb{Q})$ such that $S^{-1}AS$ is in Jordan canonical form.
It is helpful to know that a basis of the solution space of $(A-E)v = 0$ is $\left\{ \begin{bmatrix} -36 \\ 39 \\ -4 \\ 0 \\ 12 \end{bmatrix}, \begin{bmatrix} 0 \\ 17 \\ 0 \\ 4 \\ 0 \end{bmatrix} \right\}$ and a basis of the solution space of $(A-2E)v = 0$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

(b) What are the elementary divisors and the invariant factors of A?

(c) What is the minimal polynomial of A?

(d) What is the rational canonical form of A?