Proof. This is immediate, since $\mu_T(x)$ is the largest invariant factor of T, and $\chi_T(x)$ is the product of all of the invariant factors. ■

Let us make a few more remarks about the relation between the minimal polynomial and the characteristic polynomial. All of the invariant factors of T divide the minimal polynomial $\mu_T(x)$, and $\chi_T(x)$ is the product of all the invariant factors. It follows that $\chi_T(x)$ and $\mu_T(x)$ have the same irreducible factors, but with possibly different multiplicities. Since $\lambda \in K$ is a root of a polynomial exactly when $x - \lambda$ is an irreducible factor, we also have that $\chi_T(x)$ and $\mu_T(x)$ have the same roots, but with possibly different multiplicities. Finally, the characteristic polynomial and the minimal polynomial coincide precisely if V is a cyclic $K[x]$–module; i.e., the rational canonical form of T has only one block.

Of course, statements analogous to Corollary M.6.13, and of these remarks, hold for a matrix $A \in \mathrm{Mat}_n(K)$ in place of the linear transformation T.

The roots of the characteristic polynomial (or of the minimal polynomial) of $T \in \mathrm{End}_K(V)$ have an important characterization.

Definition M.6.14. We say that an *nonzero* vector $v \in V$ is an *eigenvector* of T with *eigenvalue* λ, if $Tv = \lambda v$. Likewise, we say that a nonzero vector $v \in K^n$ is an *eigenvector* of $A \in \mathrm{Mat}_n(K)$ with *eigenvalue* λ if $Av = \lambda v$.

The words "eigenvector" and "eigenvalue" are half-translated German words. The German Eigenvektor and Eigenwert mean "characteristic vector" and "characteristic value."

Proposition M.6.15. *Let $T \in \mathrm{End}_K(V)$. An element $\lambda \in K$ is a root of $\chi_T(x)$ if, and only if, T has an eigenvector in V with eigenvalue λ.*

Proof. Exercise M.6.7 ■

Exercises M.6

M.6.1. Let $h(x) \in K[x]$ be a polynomial of one variable. Show that there is a polynomial $g(x, y) \in K[x, y]$ such that $h(x) - h(y) = (x - y)g(x, y)$.

M.6.2. Set $w_j = (x - T)f_j = xf_j - \sum_i a_{i,j} f_i$. Show that $\{w_1, \dots, w_n\}$ is linearly independent over $K[x]$.

M.6.3. Verify the following assetions made in the text regarding the computation of the rational canonical form of T. Suppose that F is a free $K[x]$ module, $\Phi : F \longrightarrow V$ is a surjective $K[x]$–module homomorphism, $(y_1, \dots, y_{n-s}, z_1, \dots, z_s)$ is a basis of F, and

$$(y_1, \dots, y_{n-s}, a_1(x)z_1, \dots, a_s(x)z_s)$$

is a basis of ker(Φ). Set $v_j = \Phi(z_j)$ for $1 \le j \le s$, and

$$V_j = K[x]v_j = \text{span}(\{p(T)v_j : p(x) \in K[x]\}).$$

(a) Show that $V = V_1 \oplus \cdots \oplus V_s$.

(b) Let δ_j be the degree of $a_j(x)$. Show that $\left(v_j, Tv_j, \dots, T^{\delta_j - 1}v_j\right)$ is a basis of V_j. and that the matrix of $T_{|V_j}$ with respect to this basis is the companion matrix of $a_j(x)$.

M.6.4. Let $A = \begin{bmatrix} 7 & 4 & 5 & 1 \\ -15 & -10 & -15 & -3 \\ 0 & 0 & 5 & 0 \\ 56 & 52 & 51 & 15 \end{bmatrix}$. Find the rational canonical form of A and find an invertible matrix S such that $S^{-1}AS$ is in rational canonical form.

M.6.5. Show that χ_A is a similarity invariant of matrices. Conclude that for $T \in \text{End}_K(V)$, χ_T is well defined, and is a similarity invariant for linear transformations.

M.6.6. Since $\chi_A(x)$ is a similarity invariant, so are all of its coefficients. Show that the coefficient of x^{n-1} is the negative of the *trace* tr(A), namely the sum of the matrix entries on the main diagonal of A. Conclude that the trace is a similarity invariant.

M.6.7. Show that λ is a root of $\chi_T(x)$ if, and only if, T has an eigenvector in V with eigenvalue λ. Show that v is an eigenvector of T for some eigenvalue if, and only if, the one dimensional subspace $Kv \subseteq V$ is invariant under T.

The next four exercises give an alternative proof of the Cayley-Hamilton theorem. Let $T \in \text{End}_K(V)$, where V is n–dimensional. Assume that the field K contains all roots of $\chi_T(x)$; that is, $\chi_T(x)$ factors into linear factors in $K[x]$.

M.6.8. Let $V_0 \subseteq V$ be any invariant subspace for T. Show that there is a linear operator $\overline{T}$ on V/V_0 defined by

$$\overline{T}(v + V_0) = T(v) + V_0$$

for all $v \in V$. Suppose that $(v_1, \dots, v_k)$ is an ordered basis of V_0, and that

$$(v_{k+1} + V_0, \dots, v_n + V_0)$$

is an ordered basis of V/V_0. Suppose, moreover, that the matrix of $T_{|V_0}$ with respect to $(v_1, \dots, v_k)$ is A_1 and the matrix of $\overline{T}$ with respect to $(v_{k+1} + V_0, \dots, v_n + V_0)$ is A_2. Show that $(v_1, \dots, v_k, v_{k+1}, \dots, v_n)$ is an orderd basis of V and that the matrix of T with respect to this basis has the form

$$\begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix},$$

where B is some k–by–$(n-k)$ matrix.

M.6.9. Use the previous two exercises, and induction on n to conclude that V has some basis with respect to which the matrix of T is *upper triangular*; that means that all the entries below the main diagonal of the matrix are zero.

M.6.10. Suppose that A' is the upper triangular matrix of T with respect to some basis of V. Denote the diagonal entries of A' by $(\lambda_1, \dots, \lambda_n)$; this sequence may have repetitions. Show that $\chi_T(x) = \prod_i (x - \lambda_i)$.

M.6.11. Let $(v_1, \dots, v_n)$ be a basis of V with respect to which the matrix A' of T is upper triangular, with diagonal entries $(\lambda_1, \dots, \lambda_n)$. Let $V_0 = \{0\}$ and $V_k = \text{span}(\{v_1, \dots, v_k\})$ for $1 \le k \le n$. Show that $T - \lambda_k$ maps V_k into V_{k-1} for all k, $1 \le k \le n$. Show by induction that

$$(T - \lambda_k)(T - \lambda_{k+1}) \cdots (T - \lambda_n)$$

maps V into V_{k-1} for all k, $1 \le k \le n$. Note in particular that

$$(T - \lambda_1) \cdots (T - \lambda_n) = 0.$$

Using the previous exercise, conclude that $\chi_T(T) = 0$, the characteristic polynomial of T, evaluated at T, gives the zero transformation.

Remark M.6.16. The previous four exercises show that $\chi_T(T) = 0$, under the assumption that all roots of the characteristic polynomial lie in K. This restriction can be removed, as follows. First, the assertion $\chi_T(T) = 0$ for $T \in \text{End}_K(V)$ is equivalent to the assertion that $\chi_A(A) = 0$ for $A \in \text{Mat}_n(K)$. Let K be any field, and let $A \in \text{Mat}_n(K)$. If F is any field with $F \supseteq K$ then A can be considered as an element of $\text{Mat}_n(F)$. The characteristic polynomial of A is the same whether A is regarded as a matrix with entries in K or as a matrix with entries in F. Moreover, $\chi_A(A)$ is the same matrix, whether A is regarded as a matrix with entries in K or as a matrix with entries in F.

As is explained in Section 8.2, there exists a field $F \supseteq K$ such that all roots of $\chi_A(x)$ lie in F. It follows that $\chi_A(A) = 0$.