Mathematics 121 Review for Midterm II
April, 2005

Definitions you should know:

1. Vector space, linear independence/dependence, spanning set, basis, finite dimensional vector space, linear transformation, direct sum of vector spaces
2. Dual vector space, dual basis, matrix of a linear transformation with respect to a pair of ordered bases.
3. Similar linear transformations, similar matrices.
4. Multilinear map; symmetric, skew-symmetric, alternating multilinear map.
5. Module, free module.
6. Invariant factor decomposition and elementary divisor decomposition, for finitely generated torsion module over a PID
7. Companion matrix, rational canonical form, Jordan canonical form.

Theorems you should be able to prove:

1. A vector space with a finite spanning set has a finite basis.
2. Any two bases of a finite dimensional vector space have the same cardinality.
3. The dual space of a vector space is a vector space. If V is finite dimensional, then $\dim(V^*) = \dim(V)$ (basis dependent proof).
4. If V is finite dimensional then $V \cong V^{**}$ (basis independent proof).
5. The various isomorphism theorems for vector spaces and modules (assume the corresponding theorems for abelian groups).
6. An alternating multilinear form is skew-symmetric.
7. The determinant function is characterized as the unique alternating multilinear function of the columns of a square matrix whose value at the identity matrix is 1.
8. $\det(AB) = \det(A)\det(B)$.
9. A square matrix over a commutative ring with 1 is invertible if, and only if, its determinant is a unit.
10. Any two bases of a finitely generated free module over a commutative ring with 1 have the same cardinality.
11. If F is a finitely generated free module over a PID, then any submodule is free (the incovenient lemma).
12. If M is a finitely generated module over a PID, then every submodule is finitely generated.
13. Outline the proof of the existence of the invariant factor decomposition of a f.g. module over a PID.
14. The Cayley-Hamilton theorem.

Know how to compute:

1. **1.** Matrix of a linear transformation with respect to given bases. Change of basis for linear transformations.
2. **2.** Determinants by row reduction.
3. **3.** Rational canonical form and Jordan form for small matrices.
4. **4.** Characteristic and minimal polynomials for small matrices.
5. **5.** Possible rational canonical or Jordan forms, given the characteristic polynomial of a matrix.
6. **6.** Conversion between invariant factor and elementary divisor decompositions (say of a $K[x]$–module, or a finite abelian group).