Mathematics 121 Midterm Exam I – F. Goodman
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Version 1

Do all exercises.
Responses will be judged for accuracy, clarity and coherence.

1. Define the following:
  (a) An ideal in a ring. (You do not have to define ring.)
  (b) A prime ideal in a ring.
  (c) A unit in a ring with identity.
  (d) An irreducible element in a commutative ring with identity.
  (e) A prime element in a commutative ring with identity.
  (f) A Euclidean domain.

2. Consider the following conditions on a nonzero, nonunit element p in a **principal ideal domain** R:
- pR is a maximal ideal.
- pR is a prime ideal.
- p is prime.
- p is irreducible.

What implications hold among these conditions? Prove sufficiently many implications so that all valid implications are entailed. (If you show that A implies B and that B implies C, you don't have to tell me, or to prove, that A implies C.)

3. Prove that a Euclidean domain is a principal ideal domain.

4. Let R be any ring and I any ideal. Let n be a natural number. Denote n–by–n matrices over R by $\mathrm{Mat}_n(R)$. Show that $\mathrm{Mat}_n(I)$ is an ideal in $\mathrm{Mat}_n(R)$, and $\mathrm{Mat}_n(R)/\mathrm{Mat}_n(I) \cong \mathrm{Mat}_n(R/I)$.

5. Let R be a commutative ring with identity. Let J be an ideal in $\mathrm{Mat}_2(R)$. Define

$$J_0 = \{a \in R : \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in J\}.$$

Show that J_0 is an ideal in R and that $J = \mathrm{Mat}_2(J_0)$. Conclude that if R is a field, then $\mathrm{Mat}_2(R)$ is simple.