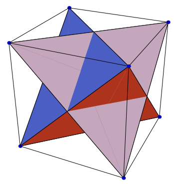

# CHAPTER M

# *Modules*

## M.1. The idea of a module

Recall that an action of a group $G$ on a set $X$ is a homomorphism

$$\varphi : G \longrightarrow \mathrm{Sym}(X).$$

Equivalently, one can view an action as a "product" $G \times X \longrightarrow X$, defined in terms of $\varphi$ by $gx = \varphi(g)(x)$, for $g \in G$ and $x \in X$. The homomorphism property of $\varphi$ translates into the mixed associative law for this product:

$$(g_1 g_2)x = g_1(g_2 x),$$

for $g_1, g_2 \in G$ and $x \in X$.

There is an analogous notion of an action of a ring $R$ on an abelian group $M$.

**Definition M.1.1.** An *action* of a ring $R$ on an abelian group $M$ is a homomorphism of $\varphi : R \longrightarrow \mathrm{End}(M)$.

Given an action $\varphi$ of $R$ on $M$, we can define a "product"

$$R \times M \longrightarrow M$$

in terms of $\varphi$ by $rm = \varphi(r)(m)$ for $r \in R$ and $m \in M$. Then the homomorphism property of $\varphi$ translates into mixed associative and distributive laws:

$$(r_1 r_2)m = r_1(r_2 m) \quad \text{and}$$
$$(r_1 + r_2)m = r_1 m + r_2 m.$$

Moreover, $\varphi(r) \in \mathrm{End}(M)$ translates into the second distributive law:

$$r(m_1 + m_2) = rm_1 + rm_2.$$

Conversely, given a product $R \times M \longrightarrow M$ satisfying the mixed associative law and the two distributive laws, for each $r \in R$, define the map $\varphi(r) : M \to M$ by $\varphi(r)(m) = rm$. Then the second distributive law says that $\varphi(r) \in \mathrm{End}(M)$ and the associative law and first distributive law say that $r \mapsto \varphi(r)$ is a ring homomorophism from $R$ to $\mathrm{End}(M)$.

**Definition M.1.2.** A module $M$ over a ring $R$ is an abelian group $M$ together with a product $R \times M \longrightarrow M$ satisfying

$$\begin{aligned} (r_1 r_2)m &= r_1(r_2 m), \\ (r_1 + r_2)m &= r_1 m + r_2 m, \quad \text{and} \\ r(m_1 + m_2) &= rm_1 + rm_2. \end{aligned}$$

**Definition M.1.3.** If the ring $R$ has identity element 1, an $R$–module $M$ is called *unital* in case $1m = m$ for all $m \in M$.

The discussion above shows that specifying an $R$–module $M$ is the same as specifying a homomorphism $\varphi$ from $R$ into the endomorphism ring of the abelian group $M$. In case $R$ has identity element 1, the $R$–module $M$ is unital if, and only if, $\varphi(1) = \mathrm{id}_M$, the identity of the ring $\mathrm{End}(M)$.

**Example M.1.4.** A unital module over a field $K$ is the same as a $K$–vector space.

**Example M.1.5.** Any ring $R$ is a module over itself (with the product $R \times R \longrightarrow R$ being the product in the ring.)

**Example M.1.6.** Any left ideal $M$ in a ring $R$ is a module over $R$ (with the product $R \times M \longrightarrow M$ being the product in the ring.)

**Example M.1.7.** For any ring $R$, and any natural number $n$, the set $R^n$ of $n$–tuples of elements of $R$ is an $R$–module with component-by-component addition and multiplication by elements of $R$.

**Example M.1.8.** Any abelian group $A$ is a unital $\mathbb{Z}$–module, with the product $\mathbb{Z} \times A \longrightarrow A$ given by $(n, a) \mapsto na =$ the $n^{th}$ power of $a$ in the abelian group $A$.

**Example M.1.9.** A vector space $V$ over a field $K$ is a module over the ring $\mathrm{End}_K(V)$, with the module action given by $Tv = T(v)$ for $T \in \mathrm{End}_K(V)$ and $v \in V$.

**Example M.1.10.** Let $T$ be a linear map defined on a vector space $V$ over a field $K$. Recall from Example 6.2.9 that there is a unital homomorphism from $K[x]$ to $\mathrm{End}_K(V)$

$$\varphi_T(\sum_i \alpha_i x^i) = \sum_i \alpha_i T^i.$$

This homomorphism makes $V$ into a unital $K[x]$–module.

Conversely, suppose $V$ is a unital $K[x]$–module, and let $\varphi : K[x] \longrightarrow \mathrm{End}(V)$ be the corresponding homomorphism. Then, $V$ is, in particular, a unital $K$–module, thus a $K$–vector space. For $\alpha \in K$ and $v \in V$, we have $\alpha v = \varphi(\alpha)(v)$. Set $T = \varphi(x) \in \mathrm{End}(V)$. We have $T(\alpha v) = \varphi(x)\varphi(\alpha)(v) = \varphi(\alpha)\varphi(x)(v) = \alpha(Tv)$ for all $\alpha \in K$ and $v \in V$. Thus $T$ is actually a linear map. Moreover, we have

$$\varphi(\sum_i \alpha_i x^i)v = \sum_i \alpha_i T^i(v),$$

so the given unital $K[x]$–module structure on $V$ is the same as the unital $K[x]$–module structure arising from the linear map $T$.

What we have called an $R$–module is also known as a *left* $R$–module. One can define a *right* $R$–module similarly.

**Definition M.1.11.** A *right module* $M$ over a ring $R$ is an abelian group $M$ together with a product $M \times R \longrightarrow M$ satisfying

$$\begin{aligned} m(r_1 r_2) &= (m r_1) r_2, \\ m(r_1 + r_2) &= m r_1 + m r_2, \quad \text{and} \\ (m_1 + m_2) r &= m_1 r + m_2 r. \end{aligned}$$

**Example M.1.12.** A right ideal $M$ in a ring $R$ is a right $R$ module.

**Example M.1.13.** Let $R$ be the ring of $n$-by-$n$ matrices over a field $K$. Then, for any $s$, the vector space $M$ of $n$-by-$s$ matrices is a left $R$ module, with $R$ acting by matrix multiplication on the left. Similarly, the vector space $N$ of $s$-by-$n$ matrices is a right $R$ module, with $R$ acting by matrix multiplication on the right.

**Convention M.1.14.** When $R$ has an identity element, we will assume, unless otherwise specified, that all $R$ modules are unital.

## Submodules

**Definition M.1.15.** Let $R$ be a ring and let $M$ be an $R$–module. An $R$–submodule of $M$ is an abelian subgroup $W$ such that for all $r \in R$ and all $w \in W$, $rw \in W$.

**Example M.1.16.** Let $R$ act on itself by left multiplication. The $R$–submodules of $R$ are precisely the left ideals of $R$.

**Example M.1.17.** Let $V$ be a vector space over $K$ and let $T \in \mathrm{End}_K(V)$ be a linear map. Give $V$ the structure of a unital $K[x]$–module as in Example M.1.10. Then the $K[x]$–submodules of $V$ are the linear subspaces $W$ of $V$ which are invariant under $T$; i.e., $T(w) \in W$ for all $w \in W$. For example, the kernel and range of $T$ are $K[x]$–submodules. The reader is asked to verify these assertions in Exercise M.1.3.

**Proposition M.1.18.** *Let $M$ be an $R$–module.*

(a) *Let $\{M_\alpha\}$ be any collection of submodules of $M$. Then $\cap_\alpha M_\alpha$ is a submodule of $M$.*
(b) *Let $M_n$ be an increasing sequence of submodules of $M$. Then $\cup_n M_n$ is a submodule of $M$.*
(c) *Let $A$ and $B$ be two submodules of $M$. Then $A + B = \{a + b : a \in A \text{ and } b \in B\}$ is a submodule of $M$.*

**Proof.** Exercise M.1.5. ■

**Example M.1.19.** Let $M$ be an $R$–module and $\mathcal{S} \subseteq M$.

(a) Define

$$R\mathcal{S} = \{r_1 s_1 + \cdots + r_n s_n : n \in \mathbb{N}, r_i \in R, s_i \in \mathcal{S}\}.$$

Then $R\mathcal{S}$ is a submodule of $M$.
(b) Let $\langle \mathcal{S} \rangle$ be the subgroup of $M$ generated by $\mathcal{S}$. Then $\langle \mathcal{S} \rangle + R\mathcal{S}$ is a submodule of $M$ containing $\mathcal{S}$.
(c) $\langle \mathcal{S} \rangle + R\mathcal{S}$ is the smallest submodule of $M$ containing $\mathcal{S}$.
(d) If $R$ has an identity element and $M$ is unital, then $\mathcal{S} \subseteq R\mathcal{S}$, and $\langle \mathcal{S} \rangle + R\mathcal{S} = R\mathcal{S}$.

The reader is asked to verify these assertions in Exercise M.1.6.

**Definition M.1.20.** $R\mathcal{S}$ is called the *submodule of $M$ generated by $\mathcal{S}$* or the *span* of $\mathcal{S}$. If $x \in M$, then $Rx = R\{x\}$ is called the *cyclic submodule generated by $x$*. If there is a finite set $\mathcal{S}$ such that $M = R\mathcal{S}$, we say that $M$ is *finitely generated*. If there is an $x \in M$ such that $M = Rx$, we say that $M$ is *cyclic*.

**Remark M.1.21.** Either $R\mathcal{S}$ or $\langle \mathcal{S} \rangle + R\mathcal{S}$ have a good claim to be called the submodule of $M$ generated by $\mathcal{S}$. Fortunately, in the case in which we are chiefly interested, when $R$ has an identity and $M$ is unital, they coincide.

## Direct Sums

**Definition M.1.22.** The *direct sum* of several $R$–modules $M_1, M_2, \dots, M_n$ is the Cartesian product endowed with the operations

$$(x_1, x_2, \dots, x_n) + (x_1', x_2', \dots, x_s') = (x_1 + x_1', x_2 + x_2', \dots, x_n + x_n')$$

and

$$r(x_1, x_2, \dots, x_n) = (rx_1, rx_2, \dots, rx_n).$$

The direct sum of $M_1, M_2, \dots, M_n$ is denoted $M_1 \oplus M_2 \oplus \cdots \oplus M_n$.

In a direct sum of $R$–modules $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$, the subset

$$\widetilde{M_i} = \{0\} \oplus \cdots \oplus M_i \oplus \cdots \oplus \{0\}$$

is a submodule isomorphic (as $R$–modules) to to $M_i$. The sum of these submodules is equal to $M$.

When is an $R$–module $M$ isomorphic to the direct sum of several $R$–submodules $A_1, A_2, \dots, A_n$? The module $M$ must be isomorphic to the direct product of the $A_i$, regarded as abelian groups. In fact, this suffices:

**Proposition M.1.23.** *Let $M$ be an $R$–module with submodules $A_1, \dots A_s$ such that $M = A_1 + \cdots + A_s$. Then the following conditions are equivalent:*

(a) $(a_1, \dots, a_s) \mapsto a_1 + \cdots + a_s$ *is a group isomorphism of* $A_1 \times \cdots \times A_s$ *onto* $M$.

(b) $(a_1, \dots, a_s) \mapsto a_1 + \cdots + a_s$ *is an $R$–module isomorphism of* $A_1 \oplus \cdots \oplus A_s$ *onto* $M$.

(c) *Each element* $x \in M$ *can be expressed as a sum*

$$x = a_1 + \cdots + a_s,$$

*with* $a_i \in A_i$ *for all* $i$*, in exactly one way.*

(d) *If* $0 = a_1 + \cdots + a_s$*, with* $a_i \in A_i$ *for all* $i$*, then* $a_i = 0$ *for all* $i$*.*

**Proof.** The equivalence of (a), (c), and (d) is by Proposition 3.3.1. Clearly (b) implies (a). On the other hand, the map $\varphi : (a_1, \dots, a_s) \mapsto a_1 + \cdots + a_s$ is actually a module homomorphism, because

$$\begin{aligned}\varphi(r(a_1, \dots, a_s)) &= \varphi((ra_1, \dots, ra_n)) = ra_1 + \cdots + ra_s \\ &= r(a_1 + \cdots a_s) = r\varphi((a_1, \dots, a_s)).\end{aligned}$$

Therefore (a) implies (b). ■

## Free modules

Let $R$ be a ring with identity element and let $M$ be a (unital) $R$–module.

We define linear independence as for vector spaces: a subset $S$ of $M$ is linearly independent over $R$ if whenever $x_1, \dots, x_n$ are *distinct* elements of $S$ and $r_1, \dots, r_n$ are elements of $R$, if

$$r_1x_1 + r_2x_2 + \cdots + r_nx_n = 0,$$

then $r_i = 0$ for all $i$.

A *basis* for $M$ is a linearly independent set $S$ with $RS = M$. An $R$ module is said to be *free* if it has a basis.

Every vector space $V$ over a field $K$ is free as a $K$–module. (We have shown this for finite dimensional vector spaces, i.e., finitely generated $K$–modules.) Modules over other rings need not be free. For example, any finite abelian group $G$ is a $\mathbb{Z}$–module, but no non-empty subset of $G$ is linearly independent; in fact, if $n$ is the order of $G$, and $x \in G$, then $nx = 0$, so $\{x\}$ is linearly dependent.

The $R$–module $R^n$ is free with the basis

$$\hat{\boldsymbol{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \hat{\boldsymbol{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \hat{\boldsymbol{e}}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

We call this the *standard basis* of $R^n$.

**Proposition M.1.24.** *Let $M$ be an $R$–module and let $x_1, \dots, x_n$ be distinct nonzero elements of $M$. The following conditions are equivalent:*

(a) *The set $F = \{x_1, \dots, x_n\}$ is a basis of $M$.*

(b) *The map*

$$(r_1, \dots, r_n) \mapsto r_1x_1 + r_2x_2 + \cdots + r_nx_n$$

*is an $R$–module isomorphism from $R^n$ to $M$.*

(c) *For each $i$, the map $r \mapsto rx_i$ is injective, and*

$$M = Rx_1 \oplus Rx_2 \oplus \cdots \oplus Rx_n.$$

**Proof.** It is easy to see that the map in (b) is an $R$–module homomorphism. The set $F$ is linearly independent if, and only if, the map is injective, and $F$ generates $M$ if, and only if the map is surjective. This shows the equivalence of (a) and (b).

Suppose that $F$ is a basis. Since $F$ spans $M$, we have $M = Rx_1 + \cdots + Rx_n$. Because $F$ is linearly independent, the sum is direct and moreover, for each $i$, the map $r \mapsto rx_i$ is injective. Thus (a) implies (c).

Suppose that (c) holds. The first condition in (c) implies that

$$(r_1, \dots, r_n) \mapsto (r_1x_1, \dots, r_nx_n)$$

is an isomorphism of $R^n$ onto the (external) direct sum

$$Rx_1 \oplus Rx_2 \oplus \cdots \oplus Rx_n.$$

Since $M$ is the direct sum of the submodules $Rx_i$, the map

$$(r_1x_1, \dots, r_nx_n) \mapsto r_1x_1 + r_2x_2 + \cdots + r_nx_n$$

is an isomorphism of the external direct sum onto $M$. Thus (c) implies (b). ■

# Exercises M.1

In the following, $R$ always denotes a ring and $M$ an $R$–module.

**M.1.1.** Show that $0x = 0$ for all $x \in M$. Here, the 0 on the left side of the equation is the zero in $R$, and the 0 on the right side is the zero in $M$.

**M.1.2.** If $R$ has an identity and $M$ is unital, show that $(-1)x = -x$ for all $x \in M$.

**M.1.3.** Prove the assertions made in Example M.1.17.

**M.1.4.** Let $I$ be a left ideal of $R$ and define

$$IM = \{r_1x_1 + \cdots + r_kx_k : k \geq 1, r_i \in I, x_i \in M\}.$$

Show that $IM$ is a submodule of $M$.

**M.1.5.** Let $N$ be a submodule of $M$. Define the annihilator of $N$ in $R$ by

$$\mathrm{ann}(N) = \{r \in R : rx = 0 \text{ for all } x \in N\}.$$

Show that $\mathrm{ann}(N)$ is a (two-sided) ideal of $R$

**M.1.6.** Prove Proposition M.1.18.

**M.1.7.** Prove the assertions made in Example M.1.19.

**M.1.8.** Show that a finite dimensional vector space $V$ over a field $K$ is not free as an $\mathrm{End}_K(V)$ module.

**M.1.9.** Let $V$ be an $n$–dimensional vector space over a field $K$. Show that $V^n$ (the direct sum of $n$ copies of $V$) is a free module over $\mathrm{End}_K(V)$.

**M.1.10.** Let $V$ be a finite dimensional vector space $V$ over a field $K$ . Let $T \in \mathrm{End}_K(V)$. Give $V$ the corresponding $K[x]$–module structure defined by $\sum_i \alpha_i x^i v = \sum_i \alpha_i T^i(v)$. Show that $V$ is not free as a $K[x]$–module.

## M.2. Homomorphisms and quotient modules

**Definition M.2.1.** Let $M$ and $N$ be modules over a ring $R$. An $R$–module *homomorphism* $\varphi : M \longrightarrow N$ is a homomorphism of abelian groups such that $\varphi(rm) = r\varphi(m)$ for all $r \in R$ and $m \in M$. An $R$–module *isomorphism* is a bijective $R$–module homomorphism. An $R$–module *endomorphism* of $M$ is an $R$–module homomorphism from $M$ to $M$.

**Notation M.2.2.** The set of all $R$–module homomorphisms from $M$ to $N$ is denoted by $\mathrm{Hom}_R(M, N)$. The set of all $R$–module endomoprhisms of $M$ is denoted by $\mathrm{End}_R(M)$.

**Example M.2.3.** Suppose $R$ is a commutative ring. For any natural number $n$, consider $R^n$ as the set of $n$-by-1 matrices over $R$ (column "vectors"). Let $T$ be a fixed $n$-by-$m$ matrix over $R$. Then left multiplication by $T$ is an $R$-module homomorphism from $R^m$ to $R^n$.

**Example M.2.4.** Fix a ring $R$. Let $T$ be a fixed $n$-by-$m$ matrix with entries in $\mathbb{Z}$. Then left multiplication by $T$ maps $R^m$ to $R^n$, and is an $R$-module homomorphism even if $R$ is non-commutative.

**Example M.2.5.** Let $R$ be the ring of $n$–by–$n$ matrices over a field. Let $M$ be the left $R$–module of $n$–by–$s$ matrices over $K$. Let $T$ be a fixed $s$–by–$s$ matrix over $K$. Then right multiplciation by $T$ is an $R$–module endomorphism of $M$.

**Proposition M.2.6.**

(a) *If $\varphi \in \mathrm{Hom}_R(M, N)$, then $\ker(\varphi)$ is a submodule of $M$ and $\varphi(M)$ is a submodule of $N$.*

(b) *$\varphi \in \mathrm{Hom}_R(M, N)$ and $\psi \in \mathrm{Hom}_R(N, P)$, then $\psi \circ \varphi \in \mathrm{Hom}_R(M, P)$.*

**Proof.** Exercise M.2.1. ■

**Proposition M.2.7.**

(a) *If $\psi, \varphi \in \mathrm{Hom}_R(M, N)$, define their sum by $(\varphi + \psi)(m) = \varphi(m) + \psi(m)$. $\mathrm{Hom}_R(M, N)$ is an abelian group under addition.*

(b) *$\mathrm{End}_R(M)$ is a ring with addition defined as above and multiplication defined by composition.*

**Proof.** Exercise M.2.2. ■

Let $M$ be an $R$–module and $N$ an $R$–submodule. We can form the quotient $M/N$ as an abelian group and considier the quotient map $\pi : M \longrightarrow M/N$ as a homomorphism of abelian groups. In fact, $M/N$ is an $R$–module and the quotient map $\pi$ is an $R$–module homomorphism.

**Proposition M.2.8.** *Let $M$ be an $R$–module and $N$ an $R$–submodule. Then the quotient $M/N$ has the structure of an $R$–module and the quotient map $\pi : M \longrightarrow M/N$ is a homomorphism of $R$–modules. If $R$ has identity and $M$ is unital, then $M/N$ is unital.*

**Proof.** We attempt to define the product of a ring element $r$ and a coset $m+N$ by the formula $r(m+N) = rm+N$. As usual, when we define an operation in terms of representatives, we have to check that the operation is well defined. If $m+N = m'+N$, then $(m-m') \in N$. Hence $rm-rm' = r(m-m') \in N$, since $N$ is a submodule. But this means that $rm+N = rm'+N$, and the operation is well defined.

Once we have checked that the action of $R$ on $M/N$ is well defined, it is easy to check that the axioms of an $R$–module are satisfied. For example, the mixed associative law is verified as follows:

$$\begin{aligned}(r_1r_2)(m+N) &= (r_1r_2)m+N = r_1(r_2m)+N \\ &= r_1(r_2m+N) = r_1(r_2(m+N)).\end{aligned}$$

The quotient map $\pi : M \longrightarrow M/N$ is a homomorphism of abelian groups, and the definition of the $R$ action on the quotient group implies that $\pi$ is an $R$–module homomorphism:

$$\pi(rm) = rm+N = r(m+N) = r\pi(m).$$

The statement regarding unital modules is also immediate from the definition of the $R$–module structure on the quotient group. ■

**Example M.2.9.** If $I$ is a left ideal in $R$, then $R/I$ is an $R$–module with the action $r(r_1+I) = rr_1+I$.

All of the homomorphism theorems for groups and rings have analogues for modules. Each of the theorems is proved by invoking the analogous theorem for abelian groups and then by checking that the homomorphisms respect the $R$–actions.

**Theorem M.2.10.** *(Homomorphism theorem for modules). Let $\varphi : M \longrightarrow P$ be a surjective homomorphism of $R$–modules with kernel $N$. Let $\pi : M \longrightarrow M/N$ be the quotient homomorphism. There is an $R$–module isomorphism $\tilde{\varphi} : M/N \longrightarrow P$ satisfying $\tilde{\varphi} \circ \pi = \varphi$. (See the following diagram.)*

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\pi"'] & P \\ M/N \arrow[ur, "\cong", "\tilde{\varphi}"'] & \end{tikzcd}$$

**Proof.** The homomorphism theorem for groups (Theorem 2.7.6) gives us an isomorphism of abelian groups $\tilde{\varphi} : M/N \to P$ satisfying $\tilde{\varphi} \circ \pi = \varphi$. We have only to verify that $\tilde{\varphi}$ also respects the $R$ actions. But this follows at once from the definition of the $R$ action on $M/N$:

$$\begin{aligned}\tilde{\varphi}(r(m+N)) &= \tilde{\varphi}(rm+N) = \varphi(rm) \\ &= r\varphi(m) = r\tilde{\varphi}(m+N).\end{aligned}$$

■

**Example M.2.11.** Let $R$ be any ring, $M$ any $R$–module, and $x \in R$. Consider the cyclic $R$–submodule $Rx$. Then $r \mapsto rx$ is an $R$–module homomorphism of $R$ onto $Rx$. The kernel of this map is called the annihilator of $x$,

$$\mathrm{ann}(x) = \{r \in R : rx = 0\}.$$

Note that $\mathrm{ann}(x)$ is a submodule of $R$, that is a left ideal. By the homomorphism theorem, $R/\mathrm{ann}(x) \cong Rx$.

**Proposition M.2.12.** *Let $\varphi : M \longrightarrow \overline{M}$ be an $R$–module homomorphism of $M$* onto *$\overline{M}$, and let $N$ denote its kernel. Then $A \mapsto \varphi^{-1}(A)$ is a bijection between $R$–submodules of $\overline{M}$ and $R$–submodules of $M$ containing $N$.*

**Proof.** By Proposition 2.7.12, $A \mapsto \varphi^{-1}(A)$ is a bijection between the subgroups of $\overline{M}$ and the subgroups of $M$ containing $N$. It remains to check that this bijection carries submodules to submodules. This is left as an exercise. ■

**Proposition M.2.13.** *Let $\varphi : M \longrightarrow \overline{M}$ be a surjective $R$–module homomorphism. Let $\overline{N}$ be a submodule of $\overline{M}$ and let $N = \varphi^{-1}(\overline{N})$. Then $m + N \mapsto \varphi(m) + \overline{N}$ is an isomorphism of $M/N$ onto $\overline{M}/\overline{N}$.*

**Proof.** Exercise M.2.5. ■

**Proposition M.2.14.** *Let $\varphi : M \longrightarrow \overline{M}$ be a surjective homomorphism of $R$–modules with kernel $N$. Let $A$ be a submodule of $M$. Then*

$$\varphi^{-1}(\varphi(A)) = A + N = \{a + n : a \in A \text{ and } n \in N\}.$$

*Moreover, $A + N$ is a submodule of $M$ containing $N$, and*

$$(A + N)/N \cong \varphi(A) \cong A/(A \cap N).$$

**Proof.** Exercise M.2.6. ■

# Exercises M.2

$R$ denotes a ring and $M$ an $R$–module.

**M.2.1.** Prove Proposition M.2.6.

**M.2.2.** Prove Proposition M.2.7.

**M.2.3.** Complete the proof of Proposition M.2.12.

**M.2.4.** Let $I$ be an ideal of $R$. Show that the quotient module $M/IM$ has the structure of an $R/I$–module.

**M.2.5.** Prove Proposition M.2.13.

**M.2.6.** Prove Proposition M.2.14.

**M.2.7.** Let $R$ be a ring with identity element. Let $M$ be a finitely generated $R$–module. Show that there is a free $R$ module $F$ and a submodule $K \subseteq F$ such that $M \cong F/K$ as $R$–modules.

## M.3. Multilinear maps and determinants

Let $R$ be a commutative ring with identity element. All $R$–modules will be assumed to be unital.

**Definition M.3.1.** Suppose that $M_1, M_2, \dots, M_n$ and $N$ are modules over $R$. A function

$$\varphi : M_1 \times \cdots \times M_n \longrightarrow N$$

is multilinear (or $R$–multilinear) if for each $j$ and for fixed elements $x_i \in M_i$ $(i \neq j)$, the map

$$x \mapsto \varphi(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_n)$$

is an $R$–module homomorphism.