Proof. Exercise M.2.6. ■

Exercises M.2

R denotes a ring and M an R–module.

M.2.1. Prove Proposition M.2.6.

M.2.2. Prove Proposition M.2.7.

M.2.3. Complete the proof of Proposition M.2.12.

M.2.4. Let I be an ideal of R. Show that the quotient module M/IM has the structure of an R/I–module.

M.2.5. Prove Proposition M.2.13.

M.2.6. Prove Proposition M.2.14.

M.2.7. Let R be a ring with identity element. Let M be a finitely generated R–module. Show that there is a free R module F and a submodule $K \subseteq F$ such that $M \cong F/K$ as R–modules.

M.3. Multilinear maps and determinants

Let R be a commutative ring with identity element. All R–modules will be assumed to be unital.

Definition M.3.1. Suppose that $M_1, M_2, \dots, M_n$ and N are modules over R. A function

$$\varphi : M_1 \times \cdots \times M_n \longrightarrow N$$

is multilinear (or R–multilinear) if for each j and for fixed elements $x_i \in M_i$ $(i \neq j)$, the map

$$x \mapsto \varphi(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_n)$$

is an R–module homomorphism.

We will be interested in the special case that all the M_i are equal. In this case we can consider the behavior of φ under permutation of the variables.

Definition M.3.2.

(a) A multilinear function $\varphi : M^n \longrightarrow N$ is said to be *symmetric* if
$$\varphi(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = \varphi(x_1, \dots, x_n)$$
for all $x_1, \dots, x_n \in M$ and all $\sigma \in S_n$.

(b) A multilinear function $\varphi : M^n \longrightarrow N$ is said to be *skew-symmetric* if
$$\varphi(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = \epsilon(\sigma)\varphi(x_1, \dots, x_n)$$
for all $x_1, \dots, x_n \in M$ and all $\sigma \in S_n$.

(c) A multilinear function $\varphi : M^n \longrightarrow N$ is said to be *alternating* if $\varphi(x_1, \dots, x_n) = 0$ whenever $x_i = x_j$ for some $i \neq j$.

Lemma M.3.3. *The symmetric group acts S_n on the set of multilinear functions from M^n to N by the formula*
$$\sigma\varphi(x_1, \dots, x_n) = \varphi(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$
The set of symmetric (resp. skew-symmetric, alternating) multilinear functions is invariant under the action of S_n.

Proof. We leave it to the reader to check that $\sigma\varphi$ is multilinear if φ is multilinear, and also that if φ is symmetric (resp. skew-symmetric, alternating), then $\sigma\varphi$ satisfies the same condition.

To check that S_n acts on Φ^n, we have to show that $(\sigma\tau)\varphi = \sigma(\tau\varphi)$. Note that
$$\sigma(\tau\varphi)(x_1, \dots, x_n) = (\tau\varphi)(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$
Now write $y_i = x_{\sigma(i)}$ for each i. Then also $y_{\tau(j)} = x_{\sigma(\tau(j))} = x_{\sigma\tau(j)}$. Thus,
$$\begin{aligned}\sigma(\tau\varphi)(x_1, \dots, x_n) &= (\tau\varphi)(y_1, \dots, y_n)\\ &= \varphi(y_{\tau(1)}, \dots, y_{\tau(n)})\\ &= \varphi(x_{\sigma(\tau(1))}, \dots, x_{\sigma(\tau(1))})\\ &= \varphi(x_{\sigma\tau(1)} \dots, x_{\sigma\tau(n)}) = (\sigma\tau)\varphi(x_1, \dots, x_n).\end{aligned}$$
■

Note that that a multilinear function is symmetric if, and only if $\sigma\varphi = \varphi$ for all $\sigma \in S_n$ and skew-symmetric if, and only if, $\sigma\varphi = \epsilon(\sigma)\varphi$ for all $\sigma \in S_n$. See Exercise xxx.

Lemma M.3.4. *An alternating multilinear function $\varphi : M^n \longrightarrow N$ is skew-symmetric.*

Proof. Fix any pair of indices $i < j$, and any elements $x_k \in M$ for k different from i, j. Define $\lambda(x, y) : M^2 \longrightarrow N$ by

$$\lambda(x, y) = \varphi(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n)$$

By hypothesis, λ is R–bilinear and alternating: $\lambda(x, x) = 0$ for all $x \in M$. Therefore,

$$\begin{aligned} 0 &= \lambda(x+y, x+y) = \lambda(x, x) + \lambda(x, y) + \lambda(y, x) + \lambda(y, y) \\ &= \lambda(x, y) + \lambda(y, x). \end{aligned}$$

Thus $\lambda(x, y) = -\lambda(y, x)$. This shows that

$$\varphi(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = (-1)\varphi(x_1, \dots, x_n)$$

when σ is the transposition (i, j).

In general, a permutation σ can be written as a product of transpositions, $\sigma = \tau_1\tau_2 \cdots \tau_\ell$. Then

$$\sigma\varphi = \tau_1(\tau_2(\cdots \tau_\ell(\varphi) \cdots)) = (-1)^\ell \varphi = \epsilon(\sigma)\varphi,$$

where we have used that S_n acts on the set of alternating multilinear functions and that ϵ is a homomorphism from S_n to $\{\pm 1\}$. ■

Lemma M.3.5. *Let $\varphi : M^n \longrightarrow N$ be a multilinear function. Then $S(\varphi) = \sum_{\sigma \in S_n} \sigma\varphi$ is a symmetric multilinear functional and $A(\varphi) = \sum_{\sigma \in S_n} \epsilon(\sigma)\sigma\varphi$ is an alternating multilinear functional.*

Proof. For $\tau \in S_n$, we have

$$\tau S(\varphi) = \sum_{\sigma \in S_n} \tau\sigma\varphi = \sum_{\sigma \in S_n} \sigma\varphi = S(\varphi),$$

since $\sigma \mapsto \tau\sigma$ is a bijection of S_n.

A similar argument shows that $A(\varphi)$ is skew-symmetric, but we have to work a little harder to show that $A(\varphi)$ is alternating.

Let $x_1, x_2, \dots, x_n \in M$, and suppose that $x_i = x_j$ for some $i < j$. The symmetric group S_n is the disjoint union of the alternating group A_n and its left coset $(i, j)A_n$, where A_n denotes the group of even permutations, and (i,j) is the transposition that interchanges i and j, and leaves all other points fixed. Thus,

$$\begin{aligned}A(\varphi)(x_1,\dots,x_n) &= \sum_{\sigma\in S_n} \epsilon(\sigma)\sigma\varphi(x_1,\dots,x_n)\\ &= \sum_{\sigma\in A_n} (\sigma\varphi(x_1,\dots,x_n) - (i,j)\sigma\varphi(x_1,\dots,x_n))\\ &= \sum_{\sigma\in A_n} (\varphi(x_{\sigma(1)},\dots,x_{\sigma(n)}) - \varphi(x_{(i,j)\sigma(1)},\dots,x_{(i,j)\sigma(n)}))\end{aligned}$$

I claim that each summand in this sum is zero.

The sequences

$$(\sigma(1),\dots,\sigma(n)) \quad \text{and} \quad ((i,j)\sigma(1),\dots,(i,j)\sigma(n))$$

are identical, except that the positions of the entries i and j are reversed. Since $x_i = x_j$, the sequences

$$(x_{\sigma(1)},\dots,x_{\sigma(n)}) \quad \text{and} \quad (x_{(i,j)\sigma(1)},\dots,x_{(i,j)\sigma(n)})$$

are identical. Therefore,

$$\varphi(x_{\sigma(1)},\dots,x_{\sigma(n)}) - \varphi(x_{(i,j)\sigma(1)},\dots,x_{(i,j)\sigma(n)}) = 0.$$

This shows that $A(\varphi)(x_1,\dots,x_n) = 0$. ■

Let $(a_1, a_2, \dots, a_n)$ be a sequence of elements of R^n. Denote the i–th entry of a_j by $a_{i,j}$. In this way, the sequence $(a_1, a_2, \dots, a_n)$ is identified with an n–by–n matrix whose j–th column is a_j. Let $\varphi : (R^n)^n \longrightarrow R$ be the multilinear function $\varphi(a_1, a_2, \dots, a_n) = a_{1,1}\cdots a_{n,n}$. Define $\Lambda = A(\varphi)$. Thus,

$$\begin{aligned}\Lambda(a_1,\dots,a_n) &= \sum_{\sigma\in S_n} \epsilon(\sigma)\varphi(a_{\sigma(1)},\dots,a_{\sigma(n)})\\ &= \sum_{\sigma\in S_n} \epsilon(\sigma)a_{1,\sigma(1)}\cdots a_{n,\sigma(n)}.\end{aligned} \tag{M.3.1}$$

According to Lemma M.3.5, Λ is an alternating multilinear function. Moreover, Λ satisfies $\Lambda(\hat{\boldsymbol{e}}_1,\dots,\hat{\boldsymbol{e}}_n) = 1$.

The summand belonging to σ in Equation M.3.1 can be written as

$$\begin{aligned}\epsilon(\sigma)\prod_{i=1}^n a_{i,\sigma(i)} &= \epsilon(\sigma)\prod_{\substack{(i,j)\\ j=\sigma(i)}} a_{i,j}\\ &= \epsilon(\sigma^{-1})\prod_{\substack{(i,j)\\ i=\sigma^{-1}(j)}} a_{i,j} = \epsilon(\sigma^{-1})\prod_{j=1}^n a_{\sigma^{-1}(j),j}\end{aligned}$$

Therefore

$$\begin{aligned}\Lambda(a_1, \dots, a_n) &= \sum_{\sigma \in S_n} \epsilon(\sigma^{-1}) a_{\sigma^{-1}(1),1} \cdots a_{\sigma^{-1}(n),n}. \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma) a_{\sigma(1),1} \cdots a_{\sigma(n),n}.\end{aligned} \tag{M.3.2}$$

Now suppose that $\mu : (R^n)^n \longrightarrow R$ is an alternating multilinear function. Let $(a_1, a_2, \dots, a_n)$ be any sequence of elements of R^n, and denote the i–th entry of a_j by $a_{i,j}$, as above, Then $a_j = \sum_i a_{i,j} \hat{\boldsymbol{e}}_i$. By the multilinearity of μ,

$$\begin{aligned}\mu(a_1, a_2, \dots, a_n) &= \mu(\sum_{j_1} a_{i_1,1} \hat{\boldsymbol{e}}_{i_1}, \dots, \sum_{i_n} a_{i_n,n} \hat{\boldsymbol{e}}_{i_n}) \\ &= \sum_{i_1, i_2, \dots, i_n} a_{i_1,1} \cdots a_{i_n,n}\, \mu(\hat{\boldsymbol{e}}_{i_1}, \dots, \hat{\boldsymbol{e}}_{i_n}).\end{aligned}$$

Because μ is alternating, $\mu(\hat{\boldsymbol{e}}_{i_1}, \dots, \hat{\boldsymbol{e}}_{i_n})$ is zero unless the sequence of indices $(i_1, \dots, i_n)$ is a permutation of $(1, 2, \dots, n)$. Thus

$$\begin{aligned}\mu(a_1, a_2, \dots, a_n) &= \sum_{\sigma \in S_n} a_{\sigma(1),1} \cdots a_{\sigma(n),n} \mu(\hat{\boldsymbol{e}}_{\sigma(1)}, \dots, \hat{\boldsymbol{e}}_{\sigma(n)}) \\ &= \sum_{\sigma \in S_n} a_{\sigma(1),1} \cdots a_{\sigma(n),n} \epsilon(\sigma) \mu(\hat{\boldsymbol{e}}_1, \dots, \hat{\boldsymbol{e}}_n) \\ &= \Lambda(a_1, \dots, a_n) \mu(\hat{\boldsymbol{e}}_1, \dots, \hat{\boldsymbol{e}}_n).\end{aligned}$$

We have proved the following result:

Proposition M.3.6. *There is a unique alternating multilinear function $\Lambda : (R^n)^n \longrightarrow R$ satisfying $\Lambda(\hat{\boldsymbol{e}}_1, \dots, \hat{\boldsymbol{e}}_n) = 1$. The function Λ satisfies*

$$\begin{aligned}\Lambda(a_1, \dots, a_n) &= \sum_{\sigma \in S_n} \epsilon(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma) a_{\sigma(1),1} \cdots a_{\sigma(n),n}.\end{aligned}$$

Moreover, if $\mu : (R^n)^n \longrightarrow R$ is alternating and multilinear, then for all $a_1, \dots, a_n \in R^n$,

$$\mu(a_1, \dots, a_n) = \Lambda(a_1, \dots, a_n) \mu(\hat{\boldsymbol{e}}_1, \dots, \hat{\boldsymbol{e}}_n).$$

Definition M.3.7. The *determinant* of an n–by–n matrix with entries in R is defined by

$$\det(A) = \Lambda(a_1, \dots, a_n),$$

where $a_1, \dots, a_n \in R^n$ are the columns of A.

Corollary M.3.8.

(a) *The determinant is characterized by the following properties:*

(i) $\det(A)$ *is an alternating multilinear function of the columns of* A.

(ii) $\det(E_n) = 1$, *where* E_n *is the* n*–by–*n *identity matrix.*

(b) *If* $\mu : \mathrm{Mat}_n(R) \longrightarrow R$ *is any function that, regarded as a function on the columns of a matrix, is alternating and multilinear, then* $\mu(A) = \det(A)\mu(E_n)$ *for all* $A \in \mathrm{Mat}_n(R)$.

Proof. This follows immediately from the properties of Λ given in Proposition M.3.6. ■

Corollary M.3.9. *Let* A *and* B *be* n*–by–*n *matrices over* R. *The determinant has the following properties*

(a) $\det(A^t) = \det(A)$, *where* A^t *denotes the transpose of* A.

(b) $\det(A)$ *is an alternating multilinear function of the rows of* A.

(c) *If* A *is a triangular matrix (i.e. all the entries above (or below) the main diagonal are zero) then* $\det(A)$ *is the product of the diagonal entries of* A.

(d) $\det(AB) = \det(A)\det(B)$

(e) *If* A *is invertible in* $\mathrm{Mat}_n(R)$, *then* $\det(A)$ *is a unit in* R, *and* $\det(A^{-1}) = \det(A)^{-1}$.

Proof. The identity $\det(A^t) = \det(A)$ of part (a) follows from the equality of the two formulas for Λ in Proposition **??**. Statement (b) follows from (a) and the properties of det as a function on the columns of a matrix.

For (c), suppose that A is lower triangular; that is the matrix entries $a_{i,j}$ are zero if $j > i$. In the expression

$$\det(A) = \sum_{\sigma \in S_n} \epsilon(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$$

the summand belonging to σ is zero unless $\sigma(i) \le i$ for all i. But the only permutation σ with this property is the identity permutation. Therefore

$$\det(A) = a_{1,1}a_{2,2}\cdots a_{n,n}.$$

To prove (d), fix a matrix A and consider the function $\mu : B \mapsto \det(AB)$. Since the columns of AB are $Ab_1, \dots, Ab_n$, where b_j is the j–th column of B, it follows that μ is an alternating mulilinear function of the columns of B. Moreover, $\mu(E_n) = \det(A)$. Therefore $\det(AB) = \mu(B) = \det(A)\det(B)$, by part (b) of the previous corollary.

If A is invertible, then

$$1 = \det(E_n) = \det(AA^{-1}) = \det(A)\det(A^{-1}),$$

so $\det(A)$ is a unit in R, and $\det(A)^{-1} = \det(A^{-1})$. ■

Lemma M.3.10. *Let $\varphi : M^n \longrightarrow N$ be an alternating mulilinear map. For any $x_1, \dots, x_n \in M$, any pair of indices $i \neq j$, and any $r \in R$,*

$$\varphi(x_1, \dots, x_{i-1}, x_i + rx_j, x_{i+1}, \dots, x_n) = \varphi(x_1, \dots, x_n).$$

Proof. Using the linearity of φ in the i—th variable, and the alternating property,

$$\begin{aligned}\varphi(x_1, \dots, x_{i-1}, &x_i + rx_j, x_{i+1}, \dots, x_n) \\ &= \varphi(x_1, \dots, x_n) + r\,\varphi(x_1, \dots, x_j, \dots, x_j, \dots, x_n) \\ &= \varphi(x_1, \dots, x_n).\end{aligned}$$

■

Proposition M.3.11. *Let A and B be n–by–n matrices over R.*

(a) *If B is obtained from A by interchanging two rows or columns, then* $\det(B) = -\det(A)$.

(b) *If B is obtained from A by multiplying one row or column of A by $r \in R$, then* $\det(B) = r\det(A)$.

(c) *If B is obtained from A by adding a multiple of one column (resp. row) to another column (resp. row), then* $\det(B) = \det(A)$.

Proof. Part (a) follows from the skew-symmetry of the determinant, part (b) from multilinearity, and part (c) from the previous lemma. ■

It is exceedingly inefficient to compute determinants by a formula involving summation over all permuations. The previous proposition provides an efficient method of computing determinants, when R is a field. One can reduce a given matrix A to triangular form by elementary row operations: interchanging two rows or adding a multiple of one row to another row. Operations of the first type change the sign of the determinant while operations of the second type leave the determinant unchanged. If B is an upper triangular matrix obtained from A in this manner, then $\det(A) = (-1)^k \det(B)$, where k is the number of row interchanges performed in the reduction. But $\det(B)$ is the product of the diagonal entries of B, by part (c) of Corollary M.3.9.

The same method works for matrices over an integral domain, as one can work in the field of fractions; of course, the determinant in the field of fractions is the same as the determinant in the integral domain.

Lemma M.3.12. *If A is a k–by–k matrix, and E_ℓ is the ℓ–by–ℓ identity matrix, then*

$$\det \begin{bmatrix} A & 0 \\ 0 & E_\ell \end{bmatrix} = \det \begin{bmatrix} E_\ell & 0 \\ 0 & A \end{bmatrix} = \det(A).$$

Proof. The function $\mu(A) = \det \begin{bmatrix} A & 0 \\ 0 & E_\ell \end{bmatrix}$ is alternating and multilinear on the columns of A, and therefore by Corollary M.3.8, $\mu(A) = \det(A)\mu(E_k)$. But $\mu(E_k) = \det(E_{k+\ell}) = 1$. This shows that $\det \begin{bmatrix} A & 0 \\ 0 & E_\ell \end{bmatrix} = \det(A)$.

The proof of the other equality is the same. ∎

Lemma M.3.13. *If A and B are square matrices, then*

$$\det \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} = \det(A)\det(B).$$

Proof. We have

$$\begin{bmatrix} A & 0 \\ C & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} E & 0 \\ C & E \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & B \end{bmatrix}.$$

Therefore, $\det \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}$ is the product of the determinants of the three matrices on th right side of the equation, by Corollary M.3.9 (d). According to the previous lemma $\det \begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix} = \det(A)$ and $\det \begin{bmatrix} E & 0 \\ 0 & B \end{bmatrix} = \det(B)$. Finally $\begin{bmatrix} E & 0 \\ C & E \end{bmatrix}$ is triangular with 1's on the diagonal, so its determinant is equal to 1, by Corollary M.3.9 (c). ∎

Let A be an n–by–n matrix over R. Let $A_{i,j}$ be the $(n-1)$–by–$(n-1)$ matrix obtained by deleting the i–th row and the j–column of A. The determinant $\det(A_{i,j})$ is called the (i, j) *minor* of A, and $(-1)^{i+j} \det(A_{i,j})$ is called the (i, j) *cofactor* of A. The matrix whose (i, j) entry is $(-1)^{i+j} \det(A_{i,j})$ is called the *cofactor matrix* of A. The transpose of the cofactor matrix is sometimes called the *adjoint matrix* of A, but this terminology should be avoided as the word adjoint has other incompatible meanings in linear algebra.

The following is called the *cofactor expansion* of the determinant.

Proposition M.3.14. *(Cofactor Expansion) Let A be an n–by–n matrix over R.*

(a) *For any i,*

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{i,j}).$$

(b) *If $i \neq k$, then*

$$0 = \sum_{j=1}^{n} (-1)^{i+j} a_{k,j} \det(A_{i,j}).$$

Proof. Fix i and j Let B_j be the matrix matrix obtained from A by replacing all the entries of the i–th row by 0's, except for the entry $a_{i,j}$, which is retained. Perform $i+j-2$ row and column interchanges to move the entry $a_{i,j}$ into the $(1, 1)$ position. The resulting matrix is

$$B'_j = \begin{bmatrix} a_{i,j} & 0 & \cdots & 0 \\ a_{1,j} & & & \\ a_{2,j} & & A_{i,j} & \\ \vdots & & & \\ a_{n,j} & & & \end{bmatrix}.$$

That is, $a_{i,j}$ occupies the $(1, 1)$ position, the remainder of the first row is zero, the remainder of the first columns contains entries from the j–th column of A, and the rest of the matrix is the square matrix $A_{i,j}$. According to Lemma M.3.13, $\det(B'_j) = a_{i,j} \det(A_{i,j})$. Therefore $\det(B_j) = (-1)^{i+j} \det(B'_j) = (-1)^{i+j} a_{i,j} \det(A_{i,j})$.

Since the matrices B_j are identical with A except in the i–th row, and the sum of the i–th rows of the B_j's is the i–th row of A, we have

$$\det(A) = \sum_j \det(B_j) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{i,j}).$$

This proves (a).

For (b), let B be the matrix that is identical to A, except that the i–th row is replaced by the k–th row of A. Since B has two identical rows, $\det(B) = 0$. Because B is the same as A except in the i–th row, $B_{i,j} = A_{i,j}$ for all j. Moreover, $b_{i,j} = a_{k,j}$. Thus,

$$0 = \det(B) = \sum_j (-1)^{i+j} b_{i,j} B_{i,j} = \sum_j (-1)^{i+j} a_{k,j} A_{i,j}.$$

■

Corollary M.3.15. *Let A be an n–by–n matrix over R and let C denote the cofactor matrix of A. Then*

$$AC^t = C^t A = \det(A)E,$$

where E denotes the identity matrix.

Proof. The sum

$$\sum_{j=1}^{n} (-1)^{i+j} a_{k,j} \det(A_{i,j}).$$

is the (k, i) entry of AC^t. Proposition M.3.14 says that this entry is equal to 0 if $k \neq i$ and equal to $\det(A)$ if $k = i$, so $AC^t = \det(A)E$.

The other equality $C^t A = \det(A)$ follows from some gymnastics with transposes: We have $(A^t)_{i,j} = (A_{j,i})^t$. Therefore,

$$(-1)^{i+j} \det((A^t)_{i,j}) = (-1)^{i+j} \det(A_{j,i}).$$

This says that the cofactor matrix of A^t is C^t. Applying the equality already obtained to A^t gives

$$A^t C = \det(A^t)E = \det(A)E,$$

and taking transposes gives

$$C^t A = \det(A)E.$$

■

Corollary M.3.16. *An element of $\mathrm{Mat}_n(R)$ is invertible if, and only if, its determinant is a unit in R.*

Proof. We have already seen that the determinant of an invertible matrix is a unit (Corollary M.3.9 (e)). On the other hand, if $\det(A)$ is a unit in R, then $\det(A)^{-1}C^t$ is the inverse of A. ■

Example M.3.17. An element of $\mathrm{Mat}_n(Z)$ has an inverse in $\mathrm{Mat}_n(\mathbb{Q})$ if its determinant is nonzero. It has an inverse in $\mathrm{Mat}_n(Z)$ if, and only if, its determinant is ± 1.

Example M.3.18. For any be an n–by–n matrix, let $\alpha(A)$ denote the transpose of the matrix of cofactors of A. I claim that

(a) $\det(\alpha(A)) = \det(A)^{n-1}$, and
(b) $\alpha(\alpha(A)) = \det(A)^{n-2}A$.

Both statements are easy to obtain under the additional assumption that R is an integral domain and $\det(A)$ is nonzero. Start with the equation $A\alpha(A) = \det(A)E$, and take determinants to get $\det(A)\det(\alpha(A)) = \det(A)^n$. Assuming that R is an integral domain and $\det(A)$ is nonzero, we can cancel $\det(A)$ to get the first assertion. Now we have $\alpha(A)\alpha(\alpha(A)) = \det(\alpha(A))E = \det(A)^{n-1}E$, as well as $\alpha(A)A = \det(A)E$. It follows that $\alpha(A)\left(\alpha(\alpha(A)) - \det(A)^{n-2}A\right) = 0$. Since $\det(A)$ is assumed to be nonzero, $\alpha(A)$ is invertible in $\mathrm{Mat}_n(F)$, where F is the field of fractions of R. Multiplying by the inverse of $\alpha(A)$ gives the second assertion.

The additional hypotheses can be eliminated by the following trick. Let $R_0 = \mathbb{Z}[x_{1,1}, x_{1,2}, \dots, x_{n,n-1}, x_{n,n}]$, the ring of polynomials in n^2 variables over $\mathbb{Z}$. Consider the matrix $X = (x_{i,j})_{1 \le i,j \le n}$ in $\mathrm{Mat}_n(R_0)$. Since R_0 is an integral domain and $\det(X)$ is nonzero in R_0, it follows that

(a) $\det(\alpha(X)) = \det(X)^{n-1}$, and
(b) $\alpha(\alpha(X)) = \det(X)^{n-2}X$.

There is a unique ring homomorphism $\varphi : R_0 \longrightarrow R$ taking 1 to 1 and $x_{i,j}$ to $a_{i,j}$, the matrix entries of A. The homomorphism extends to a homomorphism $\varphi : \mathrm{Mat}_k(R_0) \longrightarrow \mathrm{Mat}_k(R)$ for all k. By design, we have $\varphi(X) = A$.

It is easy to check that $\varphi(\det(M)) = \det(\varphi(M)$ for any square matrix M over R_0. Observe that $\varphi(M_{i,j}) = \varphi(M)_{i,j}$. Using these two observations, it follows that $\varphi(\alpha(M)) = \alpha(\varphi(M))$, and, finally, $\varphi(\det(\alpha(M))) = \det(\alpha(\varphi(M)))$.

Since $\varphi(X) = A$, applying φ to the two identities for X yield the two identities for A.

This trick is worth remembering. It is an illustration of the "principle of permanence of identities," which says that an identity that holds generically holds universally. In this instance, proving an identity for matrices with nonzero determinant over an integral domain sufficed to obtain the identity for a variable matrix over $Z[\{x_{i,j}\}]$. This in turn implied the identity for arbitrary matrices over an arbitrary commutative ring with identity.

Exercises M.3

M.3.1. Show that if $\varphi : M^n \longrightarrow N$ is multilinear, and $\sigma \in S_n$, then $\sigma\varphi$ is also multilinear. Show that each of the following sets is invariant under the action of S_n: the symmetric multilinear functions, the skew-symmetric multilinear functions, and the alternating multilinear functions.

M.3.2.

(a) Show that $(R^n)^k$ has no nonzero alternating multilinear functions with values in R, if $k > n$.

(b) Show that $(R^n)^k$ has nonzero alternating multilinear functions with values in R, if $k \leq n$.

(c) Conclude that R^n is not isomorphic to R^m as R–modules, if $m \neq n$.

M.3.3. Compute the following determinant by row reduction. Observe that the result is an integer, even though the computations involve rational numbers.

$$\det \begin{bmatrix} 2 & 3 & 5 \\ 4 & 3 & 1 \\ 3 & -2 & 6 \end{bmatrix}$$

M.3.4. Prove the cofactor expansion identity

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{i,j}).$$

by showing that the right hand side defines an alternating multilinear function of the columns of the matrix A whose value at the identity matrix is 1. It follows from Corollary M.3.8 that the right hand is equal to the determinant of A

M.3.5. Prove a cofactor expansion by columns: For fixed j,

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{i,j}).$$

M.3.6. Prove Cramer's rule: If A is an invertible n–by–n matrix over R, and $b \in R^n$, then the unique solution to the matrix equation $Ax = b$ is given by

$$x_j = \det(A)^{-1} \det(\tilde{A}_j),$$

where $\tilde{A}_j$ is the matrix obtained by replacing the j–th column of A by b.