Exercises for April 1 (ha!)

1. Let R be an integral domain, M an R–module and S a subset of R. Show that $\operatorname{ann}(S)$ is an ideal of R and $\operatorname{ann}(S) = \operatorname{ann}(RS)$.

2. Let M be a module over an integral domain R. Show that M/M_{tor} is torsion free

3. Let M be a module over an integral domain R. Suppose that $M = A \oplus B$, where A is a torsion submodule and B is free. Show that $A = M_{\text{tor}}$.

4. Let M be a torsion module over a PID R. Suppose m is a period of M and a divides M. Show that aM has period m/a.

5. Let $A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 2 & 3 & 3 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$. Let $N = \{Av : v \in \mathbb{Z}^4\}$. Then N is a subgroup of the free abelian group $\mathbb{Z}^4$, so it must also be free abelian of some rank $s \le 4$. Find a basis $\{v_1, v_2, v_3, v_4\}$ of $\mathbb{Z}^4$ and natural numbers $a_1, \dots, a_s$ with a_i dividing a_j for $i \le j$ such that $\{a_1 v_1, \dots, a_s v_s\}$ is a basis of N.

6. Let $A = \begin{bmatrix} 1-x & 0 & 2 & 4 \\ 2 & 3-x^2 & 3 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 2x & 1 \end{bmatrix}$. Find invertible matrices P, Q in $\operatorname{Mat}_4(\mathbb{Q}[x])$ and a diagonal matrix A' whose nonzero entries $a_i(x)$ satisfy $a_i(x)$ divides $a_j(x)$ for $i \le j$, such that $PAQ = A'$.

7. Find the invariant factor decomposition, the elementary divisor decomposition, and the primary decompositon of

$$\mathbb{Q}[x]/(x^9 - 8x^8 + 24x^7 - 42x^6 + 66x^5 - 78x^4 + 64x^3 - 62x^2 + 21x - 18)$$
$$\oplus\, \mathbb{Q}[x]/(x^8 - 9x^7 + 32x^6 - 62x^5 + 85x^4 - 97x^3 + 78x^2 - 44x + 24).$$

You will probably prefer to let Mathematica factor the polynomials.

8. Prove the uniqueness of the elementary divisor decomposition for a torsion module M over a PID R as follows. Suppose $M = A_1 \oplus \cdots \oplus A_s$ where each A_i is cyclic with period a power of an irreducible

(a) For any irreducible $p \in R$, and any $k \ge 1$, show that

$$p^{k-1}A_i/p^k A_i \cong \begin{cases} R/(p) & \text{if } p^k \text{ divides the period of } A_i \\ 0 & \text{otherwise.} \end{cases}$$

(b) For any irreducible $p \in R$, and any $k \ge 1$, show that

$$p^{k-1}M/p^k M \cong (R/(p))^{\ell(p,k)},$$

where $\ell(p,k)$ is the number of A_i whose period is p^α with $\alpha \ge k$.

(c) Suppose

$$\begin{aligned} M &= A_1 \oplus \cdots \oplus A_s \\ &= B_1 \oplus \cdots \oplus B_t \end{aligned}$$

where each A_i and each B_j is cyclic with period a power of an irreducible. Conclude that for each irreducible p and for each $k \geq 1$, the number of A_i with period exactly p^k is the same as the number of B_j with period exactly p^k.