Exercises for March 25

Take a look at the (recently revised) last few pages of section 3.5 on the web page, about similar matrices and similar linear transformations. Also look at the first couple of pages of the new section M.6, about invariant subspaces and block triangular or block diagonal matrices.

In the following V is an n–dimensional vector space over a field K, $T \in \mathrm{End}_K(V)$, $A \in \mathrm{Mat}_n(K)$, and E is the n–by–n identity matrix. We write $x - A$ for $xE - A$ and $x - T$ for $x\,\mathrm{id} - T$.

1. Define $\chi_A(x) = \det(x - A)$. Show that $\chi_A(x)$ is a monic polynomial of degree n. Show that $\chi_A(x)$ is a similarity invariant of matrices; i.e., it doesn't change if A is replaced by a similar matrix. We call $\chi_A(x)$ the *characteristic polynomial* of A.
2. Define $\chi_T(x) = \chi_A(x)$, where A is the matrix of T with respect to some ordered basis of V. Show that $\chi_T(x)$ is well–defined (doesn't depend on the choice of the basis) and is a similarity invariant of linear transformations. We call $\chi_T(x)$ the *characteristic polynomial* of T.
3. Since $\chi_A(x)$ is a similarity invariant, so are all of its coefficients. Show that the coefficient of x^{n-1} is the negative of the *trace* $\mathrm{tr}(A)$, namely the sum of the matrix entries on the main diagonal of A. Conclude that the trace is a similarity invariant.

Assume now that the field K contains all roots of the polynomials $\chi_A(x)$ and $\chi_A(T)$. That means that the polynomials factor as a product of n linear factors in $K[x]$. This will always be true, for example, if K is the field of complex numbers. In general, if you start out with a polynomial in $K[x]$, there is always a field $F \supseteq K$ such that the polynomial has all of its roots in F.

We say that an *nonzero* vector $v \in V$ is an *eigenvector* of T with *eigenvalue* λ, if $Tv = \lambda v$. (These are half-translated German words. The German Eigenvektor and Eigenwert mean "characteristic vector" and "characteristic value.")

4. Show that λ is a root of $\chi_T(x)$ if, and only if, T has an eigenvector in V with eigenvalue λ. Show that v is an eigenvector of T for some eigenvalue if, and only if, the one dimensional subspace $Kv \subseteq V$ is invariant under T.
5. Let $V_0 \subseteq V$ be any invariant subspace for T. Show that there is a linear operator $\overline{T}$ on V/V_0 defined by
$$\overline{T}(v + V_0) = T(v) + V_0$$
for all $v \in V$. Suppose that $(v_1, \dots, v_k)$ is an ordered basis of V_0, and that
$$(v_{k+1} + V_0, \dots, v_n + V_0)$$
is an ordered basis of V/V_0. Suppose, moreover, that the matrix of $T_{|V_0}$ with respect to $(v_1, \dots, v_k)$ is A_1 and the matrix of $\overline{T}$ with respect to $(v_{k+1} + V_0, \dots, v_n + V_0)$ is

A_2. Show that $(v_1, \ldots, v_k, v_{k+1}, \ldots, v_n)$ is an orderd basis of V and that the matrix of T with respect to this basis has the form

$$\begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix},$$

where B is some k–by–$(n-k)$ matrix.

6. Use the previous two exercises, and induction on n to conclude that V has some basis with respect to which the matrix of T is *upper triangular*; that means that all the entries below the main diagonal of the matrix are zero.

7. Suppose that A' is the upper triangular matrix of T with respect to some basis of V. Denote the diagonal entries of A' by $(\lambda_1, \ldots, \lambda_n)$; this sequence may have repetitions. Show that $\chi_T(x) = \prod_i (x - \lambda_i)$.

8. Let $(v_1, \ldots, v_n)$ be a basis of V with respect to which the matrix A' of T is upper triangular, with diagonal entries $(\lambda_1, \ldots, \lambda_n)$. Let $V_0 = \{0\}$ and $V_k = \text{span}(\{v_1, \ldots, v_k\})$ for $1 \le k \le n$. Show that $T - \lambda_k$ maps V_k into V_{k-1} for all k, $1 \le k \le n$. Show by induction that $(T - \lambda_k)(T - \lambda_{k+1}) \cdots (T - \lambda_n)$ maps V into V_{k-1} for all k, $1 \le k \le n$. Note in particular that $(T - \lambda_1) \cdots (T - \lambda_n) = 0$. Using the previous exercise, conclude that $\chi_T(T) = 0$, the characteristic polynomial of T, evaluated at T, gives the zero transformation.

9. Let $A \in \text{Mat}_n(K)$ and let $T \in \text{End}_K(K^n)$ be the linear transformation given by multiplication by A. Show that $\chi_A(A) = 0$

You have proved the theorem: $\chi_T(T) = 0$, the characteristic polynomial of T, evaluated at T, gives the zero transformation. This is under the assumption that all roots of the characteristic polynomial lie in K, but this restriction can be removed, as follows:

Let K be any field, and let $A \in \text{Mat}_n(K)$. If F is any field with $F \supseteq K$ then A can be considered as an element of $\text{Mat}_n(F)$. The characteristic polynomial of A is the same whether A is regarded as a matrix with entries in K or as a matrix with entries in F. Moreover, $\chi_A(A)$ is the same matrix, whether A is regarded as a matrix with entries in K or as a matrix with entries in F. Since there exists a field $F \supseteq K$ such that all roots of $\chi_A(x)$ lie in F, it follows that $\chi_A(A) = 0$.

Now let $T \in \text{End}_K(V)$ and let A be the matrix of T with respect to some ordred basis $\mathcal{B}$. Let $p(x) = \chi_T(x) = \chi_A(x)$. We have $p(A) = 0$. But $p(A)$ is the matrix of $p(T)$ with respect to $\mathcal{B}$, so $p(T) = 0$ as well.