Mathematics 120 Midterm Exam I – F. Goodman
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Version 1

Responses will be judged for accuracy, clarity and coherence.

1. Let N be a subgroup of a group G
  (a) Define the quotient G/N, and the quotient map $\pi : G \to G/N$.
  (b) What does it mean for N to be normal in G?
  (c) Assume that N is normal in G and show that G/N is a group (under an appropriate multiplication), and that $\pi : G \to G/N$ is a group homomorphism.

2. State and prove the Homomorphism Theorem (a.k.a. the First Isomorphism Theorem).

3. Let a and b be relatively prime natural numbers, each greater than or equal to 2.
  (a) Show that if x is an integer and both a and b divide x, then also ab divides x.
  (b) Show that the map $\theta : \mathbb{Z}_{ab} \to \mathbb{Z}_a \times \mathbb{Z}_b$ specified by $\theta([x]_{ab}) = ([x]_a, [x]_b)$ is well-defined, one-to-one, and onto.
  (c) Show that θ is a ring isomorphism.

4. **(a)** Show that every non-zero subgroup of $\mathbb{Z}$ has the form $d\mathbb{Z} = \{kd : k \in \mathbb{Z}\}$.
  (b) Show that if H is a non-zero subgroup of $\mathbb{Z}_n$, then there exists an integer d such that d divides n and $H = \{k[d] : k \in \mathbb{Z}\} = \langle [d] \rangle$.

5. Let N be a subgroup of a group G. Show that N is normal if, and only if, for each $a \in G$ there exists a $b \in G$ such that $aN = Nb$.