Mathematics 120 Final Exam – F. Goodman
December, 2004

Do any 5 problems. If you do more than 5, indicate which 5 are to be graded. Responses will be judged for accuracy, clarity and coherence. Show off, have a good time, do well.

1. State and prove the homomorphism theorem for groups, a.k.a. the first isomorphism theorem.

2. Let N be a normal subgroup of a group G and let A be another subgroup. Show that $AN = \{an : a \in A, n \in N\}$ is a subgroup of G and that $AN/N \cong A/(A \cap N)$. Under what circumstances is $AN \cong N \rtimes A$?

3. State (one form of) the fundamental theorem of finite abelian groups, a.k.a. the structure theorem for finite abelian groups. List all abelian groups of order 72.

4.

- **(a)** Show that for any abelian group, $x \mapsto x^{-1}$ is a group automorphism of order 2. In particular, $\alpha : [x] \mapsto [-x]$ is an automorphism of $\mathbb{Z}_n$ of order 2.
- **(b)** Show that $\mathbb{Z}_n \underset{\alpha}{\rtimes} \mathbb{Z}_2$ is isomorphic to the D_n (defined as symmetries of the n-gon.)
- **(c)** Determine the center of D_n. The answer is different for n even and odd.

5. What are all the quotient groups of S_4, up to isomorphism? Prove your answer. (It is permitted to just give the list of all normal subgroups of S_4 – I am now looking for the explicit identification of the quotients by these normal subgroups.)

6.

- **(a)** Show that any group G of order 28 has a normal subgroup N of order 7. Let A denote the 2-Sylow subgroup, of order 4. Show that G is the semi-direct product of N and A.
- **(b)** Show that there exists a non-abelian group of order 28 with 2-Sylow subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. In fact, $D_7 \times \mathbb{Z}_2$ is such a group.
- **(c)** Show that there exists a non-abelian group of order 28 with 2-Sylow subgroup isomorphic to Z_4. (I remind you that the automorphism group of $\mathbb{Z}_7$ is cyclic of order 6.)
- **(d)** Is $D_{14} \cong D_7 \times \mathbb{Z}_2$?

7. Use the "Burnside lemma" method to determine how many necklaces can be made with 3 amethysts (A's) and 3 black pearls (B's). Verify your answer by a common sense argument.

Extra credit: "Finish" exercise 6 by showing that there are exactly 4 groups of order 28, up to isomorphism.