REVIEW FOR FINAL

Remember that all exams are cumulative. I will likely recycle one or more questions from the previous exams. This review sheet is not meant to be exclusive. I will look at homework (including the mathematical part of computer homework), previous review sheets, previous exams, and Professor Stroyan's review material for ideas for the exam. You should too.

(1) Find parametric equations for the line segment L from $A = (1, 4, 3)$ to $B = (2, 7, -4)$.

(2) Find an equation for the plane P that is perpendicular to the line segment L in the previous exercise and which passes through the point $A = (1, 4, 3)$.

(3) Find the distance from the point $C = (4, 42)$ to the plane in exercise 2. Here is how to do this: Consider the vector $\vec{AC}$ from A to C. Project this vector on the line L, which is perpendicular to the plane. The length of the projection is the desired distance.

(4) Let $f(x, y)$ be a function of two variables with continuous partial derivatives. What is the equation of the tangent plane to the graph of $z = f(x, y)$ at a point $(x_0, y_0, z_0 = f(x_0, y_0))$?

(5) With the hypothesis of the previous exercise, what is the direction (given by a unit vector in the (x, y)–plane) of most rapid increase of the function f at the point (x_0, y_0).

(6) Continuing with the hypothesis of the previous exercise, what is a tangent vector to the graph of $z = f(x, y)$ at a point $(x_0, y_0, z_0 = f(x_0, y_0))$ which points in the direction of most rapid increase of the function f. Hint1: This was an e-exam. Hint2: the first two components of this 3D vector is the 2D vector of the previous exercise.

(7) Redo the previous three exercises with the particular function $f(x, y) = x^2y + x$ and the point $(x_0, y_0) = (1, 2)$.

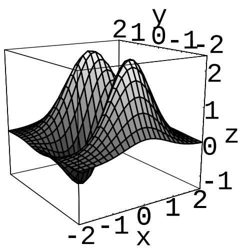
(8) Find the directional derivative of $f(x, y) = x^2y + x$ at the point $(x_0, y_0) = (1, 2)$ in the direction of the vector $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

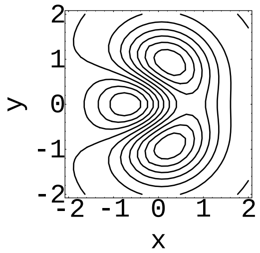
(9) Find an equation for the tangent line to the curve $x^2 + y^4 = 5$ at the point $(1, \sqrt{2})$.

(10) Find an equation for the tangent plane to the surface $x^2 + y^4 + z^2 = 6$ at the point $(1, \sqrt{2}, 1)$.

(11) Accurately state a version of the implicit function theorem, as given in lecture. Hint: look at your lecture notes.

(12) Let $f(x, y) = 3\, e^{-x^2-y^2}\,(x + 2\, y^2)$. Here are 3D and contour graphs of this function.





The partial derivatives of f are

$$\frac{\partial f}{\partial x}(x, y) = -3\, e^{-x^2-y^2} \left(-1 + 2\, x^2 + 4\, x\, y^2\right).$$

$$\frac{\partial f}{\partial y}(x, y) = -6\, e^{-x^2-y^2}\, y \left(-2 + x + 2\, y^2\right).$$

Mathematica gives the solutions to the equations

$$\frac{\partial f}{\partial x}(x, y) = 0, \quad \frac{\partial f}{\partial y}(x, y) = 0$$

as

$$\{\{x \to \frac{1}{4}, y \to \frac{-\sqrt{\frac{7}{2}}}{2}\}, \{x \to \frac{1}{4}, y \to \frac{\sqrt{\frac{7}{2}}}{2}\},$$
$$\{y \to 0, x \to -\left(\frac{1}{\sqrt{2}}\right)\}, \{y \to 0, x \to \frac{1}{\sqrt{2}}\}\}$$

Classify these critical points as maxima, minima, or neither.

(13) Find the power series for $\frac{1}{1+x^2}$ by first writing down the series for $\frac{1}{1-x}$, using this to obtain the series for $\frac{1}{1+x}$ by making an appropriate substitution, and finally using this to obtain the desired series. Show that the radius of convergence of the series is 1 (by using the test involving ratios of successive coefficients). Find the series for $\arctan(x)$ by integrating term by term.

(14) Estimate the error if the series

$$\sum_{k=0}^{\infty} \frac{\sin(7kx)}{3^k}$$

is truncated after 100 terms. That is, estimate

$$\Big|\sum_{k=101}^{\infty} \frac{\sin(7kx)}{3^k}\Big|.$$

(15) Let C be smooth curve in the plane and $\mathbf{F}(x,y) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ a vector field defined in a region containing the curve. What is the definition of $\int_C \mathbf{F} \cdot d\mathbf{x}$. Answer: Let $\mathbf{X}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $a \le t \le b$ be a smooth parametrization of C. Then

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{x} &= \int_a^b \mathbf{F}(\mathbf{X}(t)) \cdot \mathbf{X}'(t) dt \\ &= \int_a^b (f(x(t), y(t))x'(t) + g(x(t), y(t))y'(t))dt. \end{aligned}$$

(16) Let C be smooth curve in the plane and $\omega = f(x,y)dx + g(x,y)dy$ a differential one–form defined in a region containing the curve. What is the definition of $\int_C \omega$? Answer: Let $\mathbf{X}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $a \le t \le b$ be a smooth parametrization of C. Then

$$\int_C \omega = \int_a^b (f(x(t), y(t))x'(t) + g(x(t), y(t))y'(t))dt.$$

(17) Calculate $\int_C \mathbf{F} \cdot d\mathbf{x}$, where C is the straight line path from $(-3, 2)$ to $(4, 5)$ and $\mathbf{F}(x,y) = \begin{bmatrix} x^2 y \\ \sin(x) \end{bmatrix}$.

(18) Is the vector field $\mathbf{F}(x,y) = \begin{bmatrix} x^2 y \\ \sin(x) \end{bmatrix}$ conservative?

(19) State the three forms of Green's theorem (neutral differential form form, curl form, divergence form).

(20) Let C be a simple closed curve in the plane enclosing a region D. What is the connection between $\int_C x dy$ and the area of D?

(21) (Engineering project) Using the observation in the previous exercise, design a mechanical device that computes the area of a plane region when you trace a stylus around the boundary of the region. (Do not try to patent your invention, as someone beat you to it by a couple of hundred years.)